Automatica 78 (2017) 185–193

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Optimal switching for linear quadratic problem of switched systems in discrete time[☆]

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ARTICLE INFO

Article history:
Received 30 August 2015
Received in revised form
15 November 2016
Accepted 15 November 2016

Keywords:
Switched system
Lower bound dynamic system
Positive semi-definite

ABSTRACT

The optimal switching problem is attracting plenty of attention. This problem can be considered as a special type of discrete optimization problem and is NP complete. In this paper, a class of optimal switching problem involving a family of linear subsystems and a quadratic cost functional is considered in discrete time, where only one subsystem is active at each time point. By deriving a precise lower bound expression and applying the branch and bound method, a computational method is developed for solving this discrete optimization problem. Numerical examples have been implemented to demonstrate the efficiency and effectiveness of the proposed method.

1. Introduction

Switched system is an important class of hybrid systems (Yang, Cao, & Lu, 2013; Yang, Huang, & Cao, 2012). It is usually composed of multiple governed subsystems and a switching law among them. There are many real world applications of this system, such as automotive systems, electrical circuit systems, aircraft and traffic control, and so on.

In finding the optimal switching law for a switched system, the switching sequence can be sought such that a given cost functional is minimized. The characteristics of the optimal law has been studied. For example, Sussmann (1999) presented a maximum principle for hybrid optimal control problems. For a pre-specified sequence of active subsystems, Shaikh (Shaikh & Caines, 2003) proposed a class of general hybrid maximum principle. Some stability results can be found in Lin and Antsaklis (2009) and Ye and Xu (2016).

In the literature, when the sequence of the deployed system is planned in advance, there are many computational results proposed to optimize on the time when the system switches. For example, in Axelsson, Wardi, Egerstedt, and Verriest (2008), Egerstedt, Wardi, and Axelsson (2006), Giua, Seatzu, and Van Der Mee (2001) and Xu and Antsaklis (2002), the control problem was formulated as determining the optimal switching times so as to minimize a quadratic objective functional. Then the optimal switching times are sought via the derivatives of the objective functional with respect to the switching times. Furthermore, to seek both optimal switching times and optimal continuous inputs, Xu and Antsaklis (2004) presented an optimal control framework of the switched system based on a two-stage optimization. In Li, Teo, Wong, and Duan (2006), the control parameterization enhancing transform (CPET) Lee, Teo, and Jennings (1999), Liu, Loxton, and Teo (2014) and Yiu, Liu, Siu, and Ching (2010) is applied to find the optimal switching times. In Jiang, Teo, Loxton, and Duan (2012), a neighboring extremal solution is considered for a class of optimal switched impulsive control problems with perturbations, where the switching sequence is pre-specified. In this way, the problem of determining the optimal switching times for a given sequence of active subsystems can be transformed into a nonlinear programming problem which can be solved by existing gradient based techniques.

However, different from the switching time optimization which is essential continuous, the determination of the optimal switching sequence of the deployed system is combinatorial in nature. Because it is a discrete optimization problem, the search for all possible switching sequences could have an exponential complexity. In order to obtain the optimal switching sequence efficiently, Bengea and DeCarlo explored the optimal control

[☆] The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Kok Lay Teo under the direction of Editor Ian R. Petersen.

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http://dx.doi.org/10.1016/j.automatica.2016.12.002

problem of a two-switched system in Bengea and DeCarlo (2005), where the switched system was embedded into a larger family of systems. In Wardi and Egerstedt (2012), the ineffectiveness of the above method due to a possible infinite-loop procedure at each step was shown and the algorithm was improved by using other optimization techniques. At the same time, Wardi and Egerstedt proposed an adaptive-precision algorithm in Wardi, Egerstedt, and Twu (2012) which is based on simultaneous swapping of subsystems at uncountable time-sets whose Lebesgue measures are determined by the Armijo step size to solve the above problem. The similar problem is also considered in Lee and Bhattacharya (2014), where an optimal switching sequence is designed for jump linear systems with given Gaussian initial state uncertainty. Gradient-based methods can also be found in Yu, Li, and Loxton (2013) and Yu, Teo, and Bai (2013) for solving discrete-valued optimal control problem, where an equivalent penalty problem was constructed to solve the optimal control problem with piecewise constant controls and fixed switching times. Some stochastic methods such as evolutionary algorithms and simulated annealing algorithm were also developed in Cardoso, Salcedo, Azevedo, and Barbosa (1997), Costa and Oliveira (2001) and Lampinen and Zelinka (1999).

Although the continuous system is popular in practice, it should be discretized into discrete system to find the numerical solution. For example, Flaßkamp, Murphey, and Ober-Blöbaum (2012) considered the discretized hybrid dynamical systems, and proposed an optimization method to determine the optimal switching time. A linear–quadratic control problem for discrete time switched systems with uncertain subsystems is considered in Yan, Sun, and Zhu (2016) and the analytical solution is derived. However, the optimal switching problem of discrete switched system is NP-complete, and there is no literature which can consider the global solution of this problem now, even in its simple case. For this, we consider the discrete system case, where the switched subsystems are linear and the cost functional is quadratic. We aim to propose an efficient method to find the global solution of this problem.

The rest of the paper is organized as follows. In Section 2, the optimal switching problem of switched systems in discrete time is formulated, where the subsystems are linear and the cost functional is quadratic. In Section 3, we analyze the positive semi-definite property and construct a lower bound dynamic system, which is used to compute the lower bound. Then, a branch and bound method is proposed to solve this problem in Section 4. For illustration, two numerical examples are implemented in Section 5 to demonstrate the efficiency of the method.

2. Problem formulation

For a discrete time switched problem with N subsystems, we consider a dynamic system governed by the following family of linear difference equations

$$\mathbf{x}(t+1) = \mathbf{A}_i(t)\mathbf{x}(t), \quad t \in I,\ i = 1, 2, \ldots, N,$$

with initial condition

$$\mathbf{x}(0) = \mathbf{x}_0,$$

where $I = \{0, 1, \ldots, T-1\}$. For each $t \in I$, $\mathbf{A}_i(t) \in \mathbb{R}^{n\times n}$, $i = 1, 2, \ldots, N$. The state $\mathbf{x}(t)$ and initial state $\mathbf{x}_0$ are n-dimensional column vectors.

A switching sequence is denoted by a function $u : I \to \Lambda = \{1, 2, \ldots, N\}$. In particular, $u(t) = i$ means that the ith subsystem is active at time t. Let U be the set of all such switching sequences.

We formulate a discrete-time optimal switching problem as follows:

Problem 1. Find a switching sequence $u \in U$ such that

$$J(u) = \sum_{t=1}^{T} \mathbf{x}^{\mathsf{T}}(t)\mathbf{Q}(t)\mathbf{x}(t)$$

is minimized, subject to the linear dynamic constraint:

$$\mathbf{x}(t+1) = \mathbf{A}_{u(t)}(t)\mathbf{x}(t), \quad t \in I,$$
$$\mathbf{x}(0) = \mathbf{x}_0,$$

where, for each $t \in I$, $\mathbf{Q}(t+1)$ is an $n \times n$ positive semi-definite matrix, $\mathbf{A}_i(t) \in \mathbb{R}^{n\times n}$, $i \in \Lambda$. The initial state $\mathbf{x}_0$ is a given n-dimensional column vector.

Remark 1. If the linear dynamic constraint in Problem 1 is given by

$$\mathbf{x}(t+1) = \mathbf{A}_i(t)\mathbf{x}(t) + \mathbf{b}_i(t), \quad i \in \Lambda, \tag{1}$$

where for each $t \in I$ and $i \in \Lambda$, $\mathbf{A}_i(t) \in \mathbb{R}^{n\times n}$, $\mathbf{b}_i(t) \in \mathbb{R}^n$, we can solve the problem similar to Problem 1 through a suitable transformation. For this, we introduce some new symbols as follows:

$$\tilde{\mathbf{A}}_i(t) = \begin{bmatrix} \mathbf{A}_i(t) & \mathbf{b}_i(t) \\ \mathbf{0}_{1\times n} & 1 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)},$$
$$\tilde{\mathbf{Q}}(t) = \begin{bmatrix} \mathbf{Q}(t) & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)},$$
$$\mathbf{y}(t) = \begin{bmatrix} \mathbf{x}(t) \\ 1 \end{bmatrix} \in \mathbb{R}^{(n+1)}.$$

Then, the optimal switching problem with linear dynamic constraint (1) is equivalent to a new problem which is of the form Problem 1, where the cost functional is given by

$$J(u) = \sum_{t=1}^{T} \mathbf{y}^{\mathsf{T}}(t)\tilde{\mathbf{Q}}(t)\mathbf{y}(t),$$

and the dynamic constraint becomes

$$\mathbf{y}(t+1) = \tilde{\mathbf{A}}_{u(t)}(t)\mathbf{y}(t), \quad t \in I,$$
$$\mathbf{y}^{\mathsf{T}}(0) = \begin{bmatrix} \mathbf{x}_0 & 1 \end{bmatrix}^{\mathsf{T}}.$$

Remark 2. If the cost functional in Problem 1 becomes

$$J(u) = \sum_{t=1}^{T} (\mathbf{x}^{\mathsf{T}}(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{r}^{\mathsf{T}}(t)\mathbf{x}(t) + s(t)), \tag{2}$$

where for each $t \in I$, $\mathbf{Q}(t+1) \in \mathbb{R}^{n\times n}$ is positive semi-definite, $\mathbf{r}(t+1) \in \mathbb{R}^{n\times 1}$ is a column vector function, $s(t+1) \in \mathbb{R}$ is a function. This problem can also be transformed into a new problem which is of the form Problem 1 by introducing some new symbols as follows:

$$\hat{\mathbf{Q}}(t) = \begin{bmatrix} \mathbf{Q}(t) & \frac{1}{2}\mathbf{r}(t) \\ \frac{1}{2}\mathbf{r}^{\mathsf{T}}(t) & w(t) \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)},$$
$$\hat{\mathbf{A}}_i(t) = \begin{bmatrix} \mathbf{A}_i(t) & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & 1 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)},$$
$$\mathbf{y}(t) = \begin{bmatrix} \mathbf{x}(t) \\ 1 \end{bmatrix} \in \mathbb{R}^{(n+1)},$$

where $w(t)$ is any positive function such that $\hat{\mathbf{Q}}(t)$ is positive semi-definite. Then, the cost functional becomes

$$\begin{aligned}\tilde{J}(u) &= \sum_{t=1}^{T} (\mathbf{x}^{\mathsf{T}}(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{r}^{\mathsf{T}}(t)\mathbf{x}(t) + s(t)) \\ &= \sum_{t=1}^{T} \mathbf{y}^{\mathsf{T}}(t)\hat{\mathbf{Q}}(t)\mathbf{y}(t) + \sum_{t=1}^{T} (s(t) - w(t)).\end{aligned}$$