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Brief paper

Structural minimum controllability problem for switched linear continuous-time systems ☆

Sérgio Pequito, George J. Pappas

Department of Electrical and Systems Engineering, School of Engineering and Applied Science, University of Pennsylvania, United States

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ABSTRACT

This paper addresses a structural design problem in control systems, and explicitly takes into consideration the possible application to large-scale systems. More precisely, we aim to determine and characterize the minimum number of manipulated state variables ensuring structural controllability of switched linear continuous-time systems. Towards this goal, we provide a new necessary and sufficient condition that leverages both graph-theoretic and algebraic properties required to ensure feasibility of the solutions. With this new condition, we show that a solution can be determined by an efficient procedure, i.e., polynomial in the number of state variables. In addition, we also discuss the switching signal properties that ensure structural controllability and the computational complexity of determining these sequences. In particular, we show that determining the minimum number of modes that a switching signal requires to ensure structural controllability is NP-hard.

1. Introduction

Switched systems have been intensively studied, and the primary motivation comes partly from the fact that these systems have numerous applications in the control of mechanical systems, process control, automotive industry, power systems, aircraft and traffic control, and many other fields (Lin & Antsaklis, 2009; Sun, 2005). Among switched systems, those with all subsystems described by linear differential equations, referred to as *switched linear systems*, have attracted most of the attention (Lin & Antsaklis, 2009). Recent efforts aimed to analyze controllability and reachability properties of these systems (Cheng, 2005; Ji, Wang, & Guo, 2007; Sun, 2005; Sun, Ge, & Lee, 2002).

Nonetheless, only recently controllability was studied for the class of uncertain switched linear system, i.e., the parameters of subsystems' state matrices are either unknown or zero (Liu, Lin, & Chen, 2013). This assumption copes with scenarios where the system parameters are difficult to identify and obtained with a certain approximation error. Thus, structural properties that are independent of a specific value of unknown parameters are of particular interest. Subsequently, a switched linear system is said to be structurally controllable if one can find a set of values for the unknown parameters such that the corresponding switched linear system is controllable in the classical sense (Liu et al., 2013).

Motivated by economic constraints, i.e., since more actuation capabilities incur in higher cost (Olshevsky, 2014; Pequito, Kar, & Aguiar, 2016a; Pequito, Kar, & Pappas, 2015), we propose to study the *structural minimum controllability problem*, i.e., the problem of determining the smallest subset of actuated state variables ensuring structural controllability, in the context of switched linear system. Notice that understanding the allocation of actuation capabilities in large-scale systems is of fundamental importance in control systems (Skogestad, 2004; van de Wal & de Jagern, 2001). Also, such characterization is fundamental towards a better understanding of the systems resilience in case of actuation failure (Liu, Pequito, Kar, Sinopoli, & Aguiar, 2015). Towards finding the solution to the above-mentioned problem, we leverage the necessary and sufficient conditions required to assess the structural controllability of switched linear system provided in Liu et al. (2013). In other words, we complement the analysis of such systems by addressing the design problem, i.e., to optimize the actuation capabilities such that the structural controllability holds.

The structural minimum controllability problem has been fully addressed in the context of linear-time invariant (LTI) systems in Pequito et al. (2016a) for homogeneous costs; and the computational complexity analyzed for several classes of systems in Assadi, Khanna, Li, and Preciado (2015). In the current manuscript,

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E-mail addresses: sergo@seas.upenn.edu (S. Pequito), pappasg@seas.upenn.edu (G.J. Pappas).

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we extend these results to the case of switched linear continuous-time systems. Notice that whereas the solution to the design problem in LTI systems in Assadi et al. (2015) and Pequito et al. (2016a) relies on graph-theoretic properties (i.e., directed graphs interpretations of the system) of structural controllability, these no longer hold to characterize structural controllability of structural switched linear continuous-time systems, see Liu et al. (2013) for details. In particular, the analysis of structural controllability of structural switched linear continuous-time systems cannot be reduced to the analysis of a structured linear system. Consequently, in this paper we provide a systematic approach that leverages the combination of graph-theoretic and algebraic conditions to obtain and characterize the solutions to the structural minimal controllability problem for structural switched linear continuous-time system.

The current work also differs from Ramos, Pequito, Aguiar, and Kar (2015); Ramos, Pequito, Aguiar, Ramos, and Kar (2013), where the structural minimum controllability problem aimed to ensure structural controllability for each mode of the switched continuous-time linear system; note that this conservative notion contrasts with the controllability definition considered in the present manuscript. In Pequito, Kar, and Aguiar (2016b), the structural minimal controllability problem for linear time-invariant systems was considered under heterogeneous cost, i.e., the variables actuated can incur in different costs. In particular, in Pequito et al. (2016b), this problem is shown to be polynomially solvable, and, in Olshevsky (2015), the computational complexity was improved when binary costs are considered. More recently, in Pequito, Svacha, Pappas, and Kumar (2015) the problem was extended to the case where a state variable has a cost that depends on the input that actuates it, hence, leading to a multiple heterogeneous cost scenario. Alternatively, the problem of determining the minimum number of actuators from a given collection of possible actuator-state configurations was shown to be (in general) NP-hard (Pequito, Kar, & Aguiar, 2015). Notwithstanding, in Pequito et al. (2015) it was shown that the same problem can be polynomially solvable when the dynamic matrix is irreducible.

The main contributions of this paper are fourfold: (i) we provide a new necessary and sufficient condition that leverages both graph-theoretic and algebraic properties required to ensure structural controllability of switching linear continuous-time systems; (ii) we characterize the solutions to the structural minimum controllability problem for switched linear continuous-time systems. In particular, we characterize *dedicated solutions*, i.e., an actuator can only actuate a single state variable, and *minimal solutions*, i.e., the minimum number of actuators actuating the minimum number of state variables; (iii) we propose an algorithm that leverages both graph-theoretic and algebraic properties of structural controllability of switching linear continuous-time systems to determine a solution in $(mn)^\alpha$, where n denotes the number of state variables, m denotes the number of modes of the switching linear continuous-time system, and $\alpha < 2.373$ is the exponent of the $n \times n$ matrix multiplication; and (iv) since the controllability of a structural switching linear continuous-time system is tied with a particular sequence of modes the system goes through, we show that determining the minimum collection of nodes in a sequence of modes ensuring structural controllability is NP-hard.

The rest of the paper is organized as follows: Section 2 provides the formal statement of the problem addressed in this paper. Next, Section 3 reviews some concepts, introduces key results in structural systems theory and establishes their relations to graph-theoretic constructs. In Section 4, we present the main results. Next, we present an illustrative example in Section 5. Finally, Section 6 concludes the paper and discusses avenues for further research.

2. Problem statement

In this section, we formally introduce the structural minimum controllability problem for switched linear continuous-time systems.

Consider the following switched linear continuous-time system

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \tag{1}$$

where $\sigma : \mathbb{R}^+ \to \mathbb{M} \equiv \{1, \ldots, m\}$ is a switching signal, $x(t) \in \mathbb{R}^n$ the state of the system at the instant of time t, and $u(t) \in \mathbb{R}^p$ represents the piecewise continuous input signal. In the sequel, we identify (1) by the pair $(A_{\sigma(t)}, B_{\sigma(t)})$, that contains m modes with subsystems (A_i, B_i), $i \in \{1, \ldots, m\}$, and $\sigma(t) = i$ implies that the ith subsystem (A_i, B_i) is active at time instant t. Further, the switched linear continuous-time system (1) is said to be controllable (or equivalently, $(A_{\sigma(t)}, B_{\sigma(t)})$ is controllable) if for any initial state $x(0) = x_0$, and a desired state x_d, there exists a time instance $t_f > 0$, a switching signal $\sigma : [0, t_f) \to \mathbb{M}$ and an input $u : [0, t_f) \to \mathbb{R}^p$ such that $x(t_f) = x_d$. This notion of controllability enables the analysis of switching systems where we either have access to 'common' transitions and knowledge of the existing modes of the switching system, or the cases where the controller is equipped with supervisory capabilities enabling the system to switch between modes.

As previously mentioned, due to economic constraints, one is interested in deploying the minimum actuation capabilities that enable the controllability of the system, which can be captured by the following optimization problem. Given the switched linear continuous-time system (1), we aim to determine the sparsest input matrices $\{B_i^*\}_{i=1}^m$ for the different m modes required to ensure controllability, as a solution to the following optimization problem:

$$\begin{aligned} \min_{B_1,\ldots,B_m \in \mathbb{R}^{n\times n}} \quad & \sum_{i=1}^m \|B_i\|_0 \\ \text{s.t.} \quad & (A_{\sigma(t)}, B_{\sigma(t)}) \text{ is controllable,} \end{aligned} \tag{2}$$

where $\|M\|_0$ is the zero (quasi) norm, i.e., it counts the number of non-zero entries in matrix M. Notice that in (2) the matrices $B_i \in \mathbb{R}^{n\times n}$ $(i = 1, \ldots, m)$ since we do not know a priori the number of inputs required and, in the worst case scenario, by considering each matrix to be the $n \times n$ identity matrix leads to the feasibility of the problem. In other words, these dimensions are considered to ensure the problem is not ill-posed and possesses at least one solution. Subsequently, only the non-zero columns count as *effective* inputs, i.e., those required for the actuation, whereas the zero columns can be disregarded from the design procedure.

Unfortunately, the problem posed in (2) is NP-hard even when $m = 1$, see Olshevsky (2014) for details. Furthermore, the parameters associated with the linear time-invariant modes are often not accurately known, which motivates the use of structural system theory (Dion, Commault, & der Woude, 2003). Structural linear systems are linear parameterized systems with a given structure, i.e., the entries of the state space matrix are either free parameters or fixed zeros. Let $\bar{A}_{\sigma(t)} \in \{0, 1\}^{n\times n}$ denote the zero/nonzero structure or *structural pattern* of the system matrix $A_{\sigma(t)}$, whereas $\bar{B}_{\sigma(t)} \in \{0, 1\}^{n\times p}$ is the structural pattern of the input matrix $B_{\sigma(t)}$. More precisely, an entry in these matrices is zero if the corresponding entry in the system matrices is equal to zero, and described by an arbitrary parameter (denoted by one) otherwise. Therefore, a pair $(\bar{A}_{\sigma(t)}, \bar{B}_{\sigma(t)})$ is said to be structurally controllable if there exists a pair $(A'_{\sigma(t)}, B'_{\sigma(t)})$ respecting the structure of $(\bar{A}_{\sigma(t)}, \bar{B}_{\sigma(t)})$, i.e., same locations of zeros and nonzeros, such that $(A'_{\sigma(t)}, B'_{\sigma(t)})$ is controllable. Further, it can be shown that if a pair $(\bar{A}_{\sigma(t)}, \bar{B}_{\sigma(t)})$ is structurally controllable, then almost all (with respect to the Lebesgue measure) pairs with the same