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Brief paper

A new framework for solving fractional optimal control problems using fractional pseudospectral methods[☆]

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ABSTRACT

The main purpose of this work is to provide new fractional pseudospectral methods for solving fractional optimal control problems (FOCPs). We develop differential and integral fractional pseudospectral methods and prove the equivalence between them from the distinctive perspective of *Caputo fractional Birkhoff interpolation*. As a result, the present work establishes a new unified framework for solving fractional optimal control problems using fractional pseudospectral methods, which can be viewed as an extension of existing frameworks. Furthermore, we provide exact, efficient, and stable approaches to compute the associated fractional pseudospectral differentiation/integration matrices even at millions of Jacobi-type points. Numerical results on two benchmark FOCPs including a fractional bang–bang problem demonstrate the performance of the proposed methods.

1. Introduction

Fractional optimal control problems (FOCPs) can be regarded as a generalization of classical integer optimal control problems (IOCPs) in the sense that the dynamics are described by fractional differential equations (Agrawal, 2004). There are various definitions of fractional derivatives and the two most important types are the Riemann–Liouville derivatives and the Caputo derivatives. It is noteworthy here that in distinct contrast with the integer derivatives (which are locally defined in the epsilon neighborhood of a chosen point), the fractional derivatives are nonlocal in nature as they are globally defined by a definite fractional integral over a domain. Moreover, the fractional derivatives involve singular kernel/weight functions, and the solutions of fractional differential equations are usually singular near the boundaries of the domain (Chen, Shen, & Wang, 2016). More background information on the fractional calculus can be found in Oldham and Spanier (2006) and Sabatier, Agrawal, and Tenreiro Machado (2007).

Because of the complexity of most applications, FOCPs/IOCPs are often solved numerically. In recent years, a class of numerical methods called pseudospectral methods (Elnagar, Kazemi, & Razzaghi, 1995; Fahroo & Ross, 2001; Benson, Huntington, Thorvaldsen, & Rao, 2006; Huntington, 2007; Garg et al., 2010, 2011; Francolin, Benson, Hager, & Rao, 2015) has become increasingly popular in the numerical solution of IOCPs. The basic principle of pseudospectral methods is to approximate the state using a set of basis functions and discretize the dynamic constraints using collocation at a specified set of points. As a result, a continuous optimal control problem is transcribed to a finite-dimensional nonlinear programming problem (NLP) which is then solved using well-known optimization software such as SNOPT (Gill, Murray, & Saunders, 2005) and IPOPT (Biegler & Zavala, 2009). The basis functions are typically Lagrange interpolating polynomials and the collocation points are usually chosen based on Gaussian-type quadrature rules. Basically there are two primary implementation forms for pseudospectral methods: differential and integral. Although differential and integral pseudospectral methods are quite different, recent work (Tang, Liu, & Hu, 2016) has shown that they are equivalent for collocation at the Jacobi–Gauss (JG) and flipped Jacobi–Gauss–Radau (FJGR) points. Inspired by the aforementioned *global* property of the fractional derivatives and the fact of IOCPs being special cases of FOCPs, the first author

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has recently proposed the notion of fractional pseudospectral integration matrices (FPIMs) and developed integral fractional pseudospectral methods for solving FOCPs (Tang, Liu, & Wang, 2015). However, to the best of our knowledge, differential fractional pseudospectral methods for solving FOCPs have not yet received attention. Moreover, a relevant question that comes along is: does the equivalence between classical pseudospectral methods (Tang et al., 2016) still hold for fractional pseudospectral methods?

The aim of this paper is to develop new fractional pseudospectral methods and to prove the equivalence between them via a suitable Birkhoff interpolation. The present work is strikingly different from our previous work (Tang et al., 2015, 2016) in the sense of pseudospectral scheme and Birkhoff interpolation, and establishes a new unified framework for solving fractional optimal control problems using fractional pseudospectral methods. Specifically, the main contributions of this work are as follows:

(1) We propose the notion of fractional pseudospectral differentiation matrices (FPDMs) and develop differential fractional pseudospectral methods for solving FOCPs. Moreover, we propose the notion of ε-FPIMs by employing the basis of weighted Lagrange interpolating functions (Weideman & Reddy, 2000).
(2) We take a distinctive route to prove the equivalence between the proposed fractional pseudospectral methods from the perspective of *Caputo fractional Birkhoff interpolation*.
(3) We provide exact, efficient, and stable approaches to compute FPDMs/ε-FPIMs even at millions of Jacobi-type points.
(4) We extend the framework of Garg et al. (2010) to fractional pseudospectral methods with collocation at the Jacobi-type points, and that of Tang et al. (2015) to containing differential fractional pseudospectral methods.

The rest of this paper is organized as follows. In Section 2, some preliminaries are presented for subsequent developments. In Section 3, the definitions and computation of FPDMs are presented. This is followed by the definitions and computation of ε-FPIMs in Section 4. The detailed implementation of differential fractional pseudospectral methods is provided in Section 5. In Section 6, the equivalence mentioned above is proved by using the Caputo fractional Birkhoff interpolation. In Section 7, some comments on fractional pseudospectral methods are made. Numerical results on two benchmark FOCPs are shown in Section 8. Finally, Section 9 is for some concluding remarks.

2. Some preliminaries

In this section, we present the definitions of the Riemann–Liouville fractional integrals and the Caputo fractional derivatives.

Definition 1 (*Kilbas, Srivastava, & Trujillo, 2006*)**.** The left and right Riemann–Liouville fractional integrals of real order $\gamma \geq 0$ of a function $h(t)$, $t \in [t_0, t_f]$ are defined, respectively, as

$$ {}_{t_0}\mathcal{I}_t^{\gamma} h(t) \triangleq \begin{cases} \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_{t_0}^{t} (t-s)^{\gamma-1} h(s)\,\mathrm{d}s, & \gamma > 0 \\ h(t), & \gamma = 0, \end{cases} \tag{1a} $$

$$ {}_{t}\mathcal{I}_{t_f}^{\gamma} h(t) \triangleq \begin{cases} \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_{t}^{t_f} (s-t)^{\gamma-1} h(s)\,\mathrm{d}s, & \gamma > 0 \\ h(t), & \gamma = 0, \end{cases} \tag{1b} $$

where $\Gamma(\cdot)$ is the Gamma function. It is noteworthy here that for $\gamma \in \mathbb{N}$, the fractional integrals coincide with the usual iterated integrals due to the well-known Cauchy's integral formula.

Definition 2 (*Kilbas et al., 2006*)**.** The left and right Caputo fractional derivatives of real order $\gamma \in (n-1, n]$, $n = \lceil \gamma \rceil \in \mathbb{N}$ of a function $h(t) \in AC^n[t_0, t_f]$ are defined, respectively, as

$$ {}_{t_0}^{\;C}\mathcal{D}_t^{\gamma} h(t) \triangleq {}_{t_0}\mathcal{I}_t^{n-\gamma} \left(\frac{\mathrm{d}^n}{\mathrm{d}t^n} h(t) \right), \tag{2a} $$

$$ {}_{t}^{C}\mathcal{D}_{t_f}^{\gamma} h(t) \triangleq (-1)^n {}_{t}\mathcal{I}_{t_f}^{n-\gamma} \left(\frac{\mathrm{d}^n}{\mathrm{d}t^n} h(t) \right), \tag{2b} $$

where $\lceil \gamma \rceil$ denotes the smallest integer greater than or equal to γ. In particular, we have ${}_{t_0}^{\;C}\mathcal{D}_t^{0} h(t) = {}_{t}^{C}\mathcal{D}_{t_f}^{0} h(t) = h(t)$.

3. Definitions and computation of FPDMs

In this section, the definitions and computation of FPDMs are presented.

3.1. Definitions of FPDMs

Definition 3. The left and right FPDMs of real order $\gamma \in (0, 1]$ for the JG points of $\{\tau_i \in (-1, +1)\}_{i=1}^{N}$ with $-1 = \tau_0 < \tau_1 < \cdots < \tau_{N+1} = +1$ are defined, respectively, as

$$ \begin{aligned} {}_{-1}^{\;\tau}\boldsymbol{D}_{ki}^{\gamma} &\triangleq {}_{-1}^{\;C}\mathcal{D}_{\tau_k}^{\gamma} \mathcal{L}_i^{\star}(\tau) \\ &= {}_{-1}\mathcal{I}_{\tau_k}^{1-\gamma} \dot{\mathcal{L}}_i^{\star}(\tau), \\ &\quad (k = 1, 2, \ldots, N,\; i = 0, 1, \ldots, N), \end{aligned} \tag{3a} $$

$$ \begin{aligned} {}_{\tau}^{1}\boldsymbol{D}_{ki}^{\gamma} &\triangleq {}_{\tau_k}^{\;C}\mathcal{D}_{1}^{\gamma} \mathcal{L}_i^{\dagger}(\tau) \\ &= -\left({}_{\tau_k}\mathcal{I}_{1}^{1-\gamma} \dot{\mathcal{L}}_i^{\dagger}(\tau) \right), \\ &\quad (k = 1, 2, \ldots, N,\; i = 1, 2, \ldots, N+1), \end{aligned} \tag{3b} $$

where $\{\mathcal{L}_i^{\star}(\tau) \in \mathcal{P}_N\}_{i=0}^{N}$ and $\{\mathcal{L}_i^{\dagger}(\tau) \in \mathcal{P}_N\}_{i=1}^{N+1}$ are the Nth-order Lagrange interpolating polynomials associated with the interpolating points $\{\tau_i\}_{i=0}^{N}$ and $\{\tau_i\}_{i=1}^{N+1}$, respectively, defined as

$$ \mathcal{L}_i^{\star}(\tau) \triangleq \prod_{j=0, j\neq i}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad i = 0, 1, \ldots, N, \tag{4a} $$

$$ \mathcal{L}_i^{\dagger}(\tau) \triangleq \prod_{j=1, j\neq i}^{N+1} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad i = 1, 2, \ldots, N+1, \tag{4b} $$

where $\mathcal{P}_N$ denotes the set of all polynomials of degree $\leq N$. Moreover, let $\{\mathcal{L}_i(\tau) \in \mathcal{P}_{N-1}\}_{i=1}^{N}$ be the $(N-1)$th-order Lagrange interpolating polynomials associated with the interpolating points $\{\tau_i\}_{i=1}^{N}$, defined as

$$ \mathcal{L}_i(\tau) \triangleq \prod_{j=1, j\neq i}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad i = 1, 2, \ldots, N. \tag{5} $$

Then from Eqs. (4) and (5), we have

$$ \mathcal{L}_i^{\star}(\tau) = \frac{\tau - \tau_0}{\tau_i - \tau_0} \cdot \mathcal{L}_i(\tau), \quad i = 1, 2, \ldots, N, \tag{6a} $$

$$ \mathcal{L}_i^{\dagger}(\tau) = \frac{\tau_{N+1} - \tau}{\tau_{N+1} - \tau_i} \cdot \mathcal{L}_i(\tau), \quad i = 1, 2, \ldots, N. \tag{6b} $$

Note that Eq. (6a) has already been given in Tang et al. (2016, Eq. (11)).

Definition 4. The left FPDM of real order $\gamma \in (0, 1]$ for the FJGR points of $\{\hat{\tau}_i \in (-1, +1]\}_{i=1}^{N}$ with $-1 = \hat{\tau}_0 < \hat{\tau}_1 < \cdots < \hat{\tau}_N = +1$, and the right FPDM of real order $\gamma \in (0, 1]$ for the