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Delay-independent incremental stability in time-varying monotone systems satisfying a generalized condition of two-sided scalability[☆]

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ABSTRACT

Monotone systems generated by delay differential equations with explicit time-variation are of importance in the modeling of a number of significant practical problems, including the analysis of communications systems, population dynamics, and consensus protocols. In such problems, it is often of importance to be able to guarantee delay-independent incremental asymptotic stability, whereby all solutions converge toward each other asymptotically, thus allowing the asymptotic properties of all trajectories of the system to be determined by simply studying those of some particular convenient solution. It is known that the classical notion of quasimonotonicity renders time-delayed systems monotone. However, this is not sufficient alone to obtain such guarantees. In this work we show that by combining quasimonotonicity with a condition of scalability motivated by wireless networks, it is possible to guarantee incremental asymptotic stability for a general class of systems that includes a variety of interesting examples. Furthermore, we obtain as a corollary a result of guaranteed convergence of all solutions to a quantifiable invariant set, enabling time-invariant asymptotic bounds to be obtained for the trajectories even if the precise values of time-varying parameters are unknown.

1. Introduction

Monotone systems represent an important class of dynamical systems that are of interest both for their applicability to a number of practical problems and for their rich mathematical structure. The order-preserving structure of these systems allows strong results about their stability properties to be obtained. In the celebrated work (Hirsch, 1988), Hirsch established results of generic convergence, guaranteeing convergence of almost every bounded solution of any time-invariant system for which the monotonicity property holds strongly to the equilibrium set, provided this set is nonempty. In systems of differential equations that are not autonomous, however, the equilibrium set of even a monotone system will frequently be empty, so the generic convergence results are not directly applicable. Instead, a property that is often of interest is the concept of incremental asymptotic stability. The system is said to be incrementally asymptotically stable, in some specified set of initial conditions, if all solutions starting within this set converge to each other uniformly, and so this idea is of particular use in problems of system tracking or prediction. This concept, proposed in Angeli (2002) and often referred to simply as incremental stability, was seen in Rüffer, van de Vouw, Mueller (2013) to be closely linked to the notions of convergence and contraction studied in Demidovich (1967) and Lohmiller and Slotine (1998) respectively, and has been investigated in numerous interesting works such as Forni and Sepulchre (2014), Fromion, Monaco, and Normand-Cyrot (1996), Fromion and Scorletti (2002), Jouffroy and Fossen (2010), Pavlov, Pogromsky, van de Wouw, Nijmeijer (2004) and Sontag and Ingalls (2002). Two recent papers in which incremental asymptotic stability in general nonlinear systems with time-delays has been studied are Chaillet, Pogromsky, and Rüffer (2013) and Pola, Pepe, Di Benedetto, and Tabuada (2010). Pola et al. (2010) presented a Lyapunov–Krasovskii framework for verifying incremental

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asymptotic stability in delayed systems, while Chaillet et al. (2013) proposed an incremental formulation of Lyapunov–Razumikhin approaches and used this to formulate sufficient algebraic conditions for incremental asymptotic stability in Lur'e systems.

Monotone systems have been extensively studied in the context of a variety of applications, including the motivating problem of population modeling (Hirsch, 1982), the analysis of biological systems (Hsu & Waltman, 1992), and the notion of antagonistic consensus (Altafini, 2013). One particular area in which they have received significant attention is in the control of antenna uplink powers in wireless networks. This follows from the seminal paper (Yates, 1995), in which it was shown that a general class of such power control algorithms can be modeled in discrete-time by a general monotone system satisfying a condition of scalability. Stability of a continuous-time version of this framework with time-varying delays was considered in Lestas (2012). Stability issues in general classes of delayed autonomous monotone systems have been addressed in the literature, e.g. in Bokharaie, Mason, and Verwoerd (2010), Devane and Lestas (2015) and Smith (1995). It was shown in Devane and Lestas (2014) that when explicit time-variation is incorporated within the class of systems considered in Lestas (2012), versions of the monotonicity and scalability properties introduced in Yates (1995) can be sufficient to guarantee that all trajectories converge to one another, independent of arbitrary bounded time-varying delays. However, the notion of monotonicity used in the analysis is significantly stronger than the classical property of quasimonotonicity that is needed to specify a general monotone system of delay differential equations, thus restricting its applicability to wider classes of systems. As such, it is desirable to investigate whether these approaches can be generalized to yield conclusions guaranteeing delay-independent incremental asymptotic stability in general monotone systems under a condition inspired by the property of scalability.

Within this paper we will demonstrate that this is possible, proving results guaranteeing incremental asymptotic stability for a broad class of time-varying nonlinear systems encompassing a range of interesting practical examples. Moreover, we shall in fact see that, similarly to the extensions provided to the wireless network analysis in Sung and Leung (2005), this framework can be further generalized by combining the quasimonotonicity and scalability conditions into a single, weaker property of uniform two-sided quasiscalability, thereby yielding conclusions of delay-independent incremental asymptotic stability even for classes of systems that may not be monotone. As a corollary, we also show that monotonicity properties can be exploited in this context to deduce convergence of all solutions to a bounded invariant set, thus allowing time-invariant asymptotic bounds for the trajectories to be obtained.

The paper is structured as follows. We begin in Section 2 by reviewing some of the theory concerning stability and convergence of systems of delay differential equations. We present in Section 3.1 the general framework of strictly positive monotone systems of delay differential equations within which our analysis will take place. In Section 3.2, we formulate an assumption of uniform two-sided quasiscalability, and we show that this condition alone can be used to deduce incremental asymptotic stability whenever the system admits a solution satisfying a particular boundedness condition. Section 3.3 then presents a constraint on the system's explicit time-variation that allows us to guarantee the existence of a bounded invariant set. These results are then combined, in Section 3.4, to yield our main result of guaranteed delay-independent incremental asymptotic stability for systems for which both of the foregoing assumptions are satisfied. Various applications are given in Section 4, and finally conclusions are drawn in Section 5.

2. Preliminaries

2.1. Notation

Within the paper, we will use $\mathbb{R}_+$ to denote the set of nonnegative real numbers $\{s \in \mathbb{R}: s \geq 0\}$ and $\mathbb{R}_{++}$ to denote the set of positive reals $\{s \in \mathbb{R}: s > 0\}$. In the preliminaries, the notation $\|\cdot\|$ can represent any norm on $\mathbb{R}^n$, however, within our analysis we will work mainly with the infinity norm, denoted by $\|\cdot\|_\infty$. Inequalities in $\mathbb{R}^n$ are defined as follows: $x \geq y$ means $x_i \geq y_i$ for all i, $x > y$ means $x_i \geq y_i$ for all i and $x \neq y$, and $x \gg y$ means $x_i > y_i$ for all i. We will use $\mathcal{C}([a, b], \Omega)$ to denote the Banach space of continuous functions mapping $[a, b] \subseteq \mathbb{R}$ into $\Omega \subseteq \mathbb{R}^n$, with elemental norm $\|\phi\|_{\mathcal{C}} = \sup_{a \leq \theta \leq b} \|\phi(\theta)\|$. Inequalities in $\mathcal{C}([a, b], \Omega)$ are treated pointwise, e.g. $\phi \geq \psi$ means $\phi(\theta) \geq \psi(\theta)$ for all $\theta \in [a, b]$.

2.2. Background theory

We wish to investigate the long-term behavior of solutions of a general class of nonautonomous systems of differential equations with arbitrary bounded time-delays. A detailed study of the theory of such systems can be found in Hale and Lunel (1993). Following this framework, if $x \in \mathcal{C}([t_0 - r, t_0 + A], \Omega)$ for a given $t_0 \geq 0$ and $A > 0$, we define the segment $x_t \in \mathcal{C}([-r, 0], \Omega)$ of x as $x_t(\theta) = x(t + \theta)$ for all $\theta \in [-r, 0]$, for any $t \in [t_0, t_0 + A]$. The general delay differential equation can then be written as

$$\dot{x}(t) = f(t, x_t), \tag{1}$$

where $f : \mathbb{R}_+ \times \mathcal{C}([-r, 0], \Omega) \to \mathbb{R}^n$ is assumed to be continuous in its first argument and to satisfy a local Lipschitz property, uniformly in t, in its second argument[1], which ensure existence and uniqueness of solutions and their continuous dependence on the initial data (Hale & Lunel, 1993, Theorems 2.2.1, 2.2.2, and 2.2.3). The function x is then said to be a solution of (1) through (t_0, ϕ) for the initial condition $\phi \in \mathcal{C}([-r, 0], \Omega)$ if $x(t)$ satisfies (1) for all $t \geq t_0$ and $x_{t_0} = \phi$ on $[-r, 0]$. When explicit consideration of the initial conditions is required, this solution can be denoted by $x(t, t_0, \phi)$, with the corresponding delayed segment written as $x_t(t_0, \phi)$.

In this paper, we wish to investigate when it is possible to guarantee certain properties of stability and convergence for the system (1). We now define a standard class of comparison functions in terms of which the relevant notions can be formulated. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if

- For each fixed $\tilde{s} \geq 0$, $\beta(0, \tilde{s}) = 0$ and $\beta(s, \tilde{s})$ is strictly increasing in s,
- For each fixed $s > 0$, $\beta(s, \tilde{s})$ is decreasing in $\tilde{s}$ and $\lim_{\tilde{s} \to \infty} \beta(s, \tilde{s}) = 0$.

We first introduce a notion of uniform asymptotic stability with respect to a particular trajectory of (1).

Definition 1. The solution $X(t)$ of (1) is uniformly asymptotically stable in a set of initial conditions $S \subseteq \mathcal{C}([-r, 0], \Omega)$ if there exists a function $\beta \in \mathcal{KL}$ such that for all $t_0 \geq 0$, all $\phi \in S$, and all $t \geq t_0$,

$$\|x_t(t_0, \phi) - X_t\|_{\mathcal{C}} \leq \beta\left(\|\phi - X_{t_0}\|_{\mathcal{C}}, t - t_0\right).$$

It is globally uniformly asymptotically stable if the set S can be chosen as the entire space $\mathcal{C}([-r, 0], \Omega)$.

[1] By this we mean that for any compact subset $\Phi \subseteq \Omega$, there exists a constant $L \geq 0$ such that $\|f(t, \phi) - f(t, \psi)\| \leq L\|\phi - \psi\|_{\mathcal{C}}$ for all $\phi, \psi \in \mathcal{C}([-r, 0], \Phi)$ and all $t \geq 0$.