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Explicit model predictive control: A connected-graph approach[☆]

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ABSTRACT

The ability to solve model predictive control (MPC) problems of linear time-invariant systems explicitly and offline *via* multi-parametric quadratic programming (mp-QP) has become a widely used methodology. The most efficient approaches used to solve the underlying mp-QP problem are either based on combinatorial considerations, which scale unfavorably with the number of constraints, or geometrical considerations, which require heuristic tuning of the step-size and correct identification of the active set. In this paper, we describe a novel algorithm which unifies these two types of approaches by showing that the solution of a mp-QP problem is given by a connected graph, where the nodes correspond to the different optimal active sets over the parameter space. Using an extensive computational study, as well as the explicit MPC solution of a combined heat and power system, the merits of the proposed algorithm are clearly highlighted.

1. Introduction

Model Predictive Control (MPC) is a powerful, model-based control paradigm, which enables the optimal control of multivariable, constrained systems (Bahakim & Ricardez Sandoval, 2014; Chawankul, Ricardez Sandoval, Budman, & Douglas, 2007; Heidarinejad, Liu, & Christofides, 2013; Mastragostino, Patel, & Swartz, 2014; Mhaskar, El-Farra, & Christofides, 2005; Nikandrov & Swartz, 2009; Soliman, Swartz, & Baker, 2008; Trifkovic, Sheikhzadeh, Choo, & Rohani, 2012). At its heart is thereby the idea of a receding horizon approach, where the optimization problem associated with the MPC problem is solved online at each time step (Rawlings & Mayne, 2009). One way to avoid this online computational step is the explicit solution of the MPC problem *via* its reformulation into an equivalent multi-parametric quadratic programming (mp-QP) problem, which is solved once and offline (Bemporad, Morari, Dua, & Pistikopoulos, 2002; Mayne & Raković, 2002; Pistikopoulos et al., 2015); also used in many applications (see e.g. Axehill, Besselmann, Raimondo, & Morari, 2014; Feller, Johansen, & Olaru, 2013; Kouramas, Faísca, Panos, & Pistikopoulos, 2011; Kouramas, Panos, Faísca, & Pistikopoulos, 2013; Krieger & Pistikopoulos, 2014; Mayne, Raković, & Kerrigan, 2007; Oberdieck & Pistikopoulos, 2015; Pistikopoulos, 2009; Pistikopoulos, 2012; Pistikopoulos & Diangelakis, 2016; Rivotti & Pistikopoulos, 2014a; Rivotti & Pistikopoulos, 2014b; Wen, Ma, & Ydstie, 2009).

Due to the reformulation into a mp-QP problem, the application of explicit MPC is intimately linked to the ability to solve mp-QP problems. The most efficient solution procedures to date can broadly be classified into geometrical and combinatorial approaches. In the geometrical approaches, the region where a certain active set is optimal is used as the basis for the algorithm, with the focus then being laid on exploring the parameter space by moving from one region to another (Baotic, 2002; Bemporad, 2015; Bemporad, Morari et al., 2002; Dua, Bozinis, & Pistikopoulos, 2002; Spjøtvold, Kerrigan, Jones, Tøndel, & Johansen, 2006; Tøndel, Johansen, & Bemporad, 2003). On the other hand, combinatorial approaches are based on the optimal active set itself, and then an evaluation of all feasible combinations of active sets *via* a suitable branch-and-bound approach (Feller et al., 2013; Gupta, Bhartiya, & Nataraj, 2011; Herceg, Jones, Kvasnica, & Morari, 2015).

In this paper we use the ability to infer the optimal active set of adjacent critical regions (Tøndel et al., 2003) to show that the solution of mp-QP problems, and thus of explicit MPC problems, is given by a connected graph. This property is used to devise a novel solution algorithm, which uses the concept of a connected graph in a combinatorial setting, where the powerful fathoming criterion of Gupta et al. (2011) is still applicable. The merit of this new algorithm is shown in an extensive computational study featuring

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test sets of 100 mp-LP and mp-QP problems of various size each, as well as in the explicit model predictive control of a combined heat-and-power (CHP) system, where the scalability of the novel approach is highlighted by considering an increasing control and output horizon.

1.1. Nomenclature

We denote $0_{n\times m}$ as the zero matrix of dimension $n \times m$. Let $a \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, then a_k and A_k denote the kth element and row of a and A, respectively, and A_{kj} denotes the element in the k-row and jth column of A. Additionally, $|a|$ denotes the cardinality of the set a. Let $n, k \in \mathbb{R}$ and p be a set. Then the binomial coefficient is denoted as $\binom{n}{k}$, while $\binom{p}{k}$ denotes the set of all possible sets of cardinality k which are subsets of p. Lastly, let P be a polytope, then $\mathrm{int}\,(P)$ denotes the interior of P.

2. Theoretical background

2.1. From a MPC to a mp-QP problem

The general MPC problem formulation is considered as follows:

$$\begin{aligned}\min_{u}\quad & x_N^T P x_N \\ & + \sum_{k=1}^{N-1}\left(x_k^T Q_k x_k + \left(y_k - y_k^R\right)^T QR_k \left(y_k - y_k^R\right)\right) \\ & + \sum_{k=0}^{M-1}\left(\left(u_k - u_k^R\right)^T R_k \left(u_k - u_k^R\right) + \Delta u_k^T R1_k \Delta u_k\right) \\ \text{s.t.}\quad & x_{k+1} = A\,x_k + B\,u_k + C\,d_k \\ & y_k = D\,x_k + E\,u_k + e \\ & u_{\min} \le u_k \le u_{\max} \\ & \Delta u_{\min} \le \Delta u_k \le \Delta u_{\max} \\ & x_{\min} \le x_k \le x_{\max} \\ & y_{\min} \le y_k \le y_{\max},\end{aligned} \tag{1}$$

where x_k are the state variables, u_k and u_k^R are the control variables and their respective set points, Δu_k denotes the difference between two consecutive control actions, y_k and y_k^R are the outputs and their respective set points, d_k are the measured disturbances, Q_k, R_k, $R1_k$ and QR_k are their corresponding weights in the quadratic objective function, P is the stabilizing term derived from the Riccati Equation for discrete systems, N and M are the output horizon and control horizon respectively, k is the time step, A, B, C, D, E are the matrices of the discrete linear state space model and e denotes the mismatch between the actual system output and the predicted output at initial time.

This quadratic programming (QP) problem is reformulated into a mp-QP problem according to the principles presented in Bemporad, Morari et al. (2002). The correlation of the state variables at time step k to the uncertain parameters and the control variables u_k is derived by the reverse substitution of the states in the linear state space model. The states at the initial time (x_0), the set points (u_k^R and y_k^R), the initial output mismatch, the previous control actions in Δu_k and the disturbances (d_k) are treated as uncertain parameters denoted by the parameter vector θ. The general form of the mp-QP is presented in problem (2):

$$\begin{aligned} z(\theta) = \min_{x}\quad & (Qx + H\theta + c)^T x \\ \text{s.t.}\quad & Ax \le b + F\theta \\ & \theta \in \Theta := \{\theta \in \mathbb{R}^q \,|\, CR_A\theta \le CR_b\},\end{aligned} \tag{2}$$

where $Q \in \mathbb{R}^{n\times n}$ is symmetric positive definite, $H \in \mathbb{R}^{n\times q}$, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $F \in \mathbb{R}^{m\times q}$, $CR_A \in \mathbb{R}^{r\times q}$, $CR_b \in \mathbb{R}^r$ and Θ is compact. It should be noted that in this case x is a vector that consists of M control actions, hence the lack of k to denote the time step. The letters n and q in problem (2) denote the size of the control and parameter vector respectively, consequently the size of the parametric solution and parametric space of the problem.

Remark 1. In the case where $Q = 0_{n\times n}$ and $H = 0_{n\times q}$, problem (2) only features the linear objective $c^T x$ and thus becomes a multi-parametric linear programming (mp-LP) problem, a class of problems which has also been used in the realm of linear explicit MPC (Barić, Jones, & Morari, 2006; Bemporad, Borrelli, & Morari, 2000, 2002; Jones, Barić, & Morari, 2007; Jones, Kerrigan, & Maciejowski, 2007).

2.2. Multi-parametric quadratic programming

The solution of problem (2) has been studied extensively in the literature and is summarized below (Bemporad, Morari et al., 2002; Dua et al., 2002; Gupta et al., 2011; Herceg et al., 2015; Spjøtvold et al., 2006; Tøndel et al., 2003):

Definition 2 (*Critical Regions Bemporad, Morari et al., 2002*)**.** A function $x(\theta) : \Theta \mapsto \mathbb{R}^n$, where $\Theta \in \mathbb{R}^q$ is a polyhedral set, is piecewise affine if it is possible to partition Θ into convex polyhedral regions, CR_i, and

$$x(\theta) = K_i\theta + r_i, \quad \forall\theta \in CR_i, \tag{3}$$

where CR_i is referred to as the ith critical region. Note that piecewise quadratic functions are defined analogously.

Theorem 3 (*mp-LP and mp-QP Solutions Bemporad, Morari et al., 2002; Dua et al., 2002*)**.** *Consider the mp-QP problem (2) and let Q be positive definite. Then the set of feasible parameters $\Theta_f \subseteq \Theta$ is convex, the optimizer $x(\theta) : \Theta_f \mapsto \mathbb{R}^n$ is continuous and piecewise affine, and the optimal solution $z(\theta) : \Theta_f \mapsto \mathbb{R}$ is continuous, and piecewise quadratic. In the case of mp-LP problems, the optimal objective function $z(\theta) : \Theta_f \mapsto \mathbb{R}$ is continuous, convex and piecewise affine.*

Additionally, we state the following lemma:

Lemma 4 (*Active Set Representation Gupta et al., 2011*)**.** *Each critical region is uniquely defined by the optimal active set associated with it, and the solution to problem (2) can be represented as the set of all optimal active sets.*

For the solution of problem (2), three different approaches have been considered extensively.[1]

2.3. The geometrical approach

Since the critical regions are polytopes, the key idea is to explore the parameter space geometrically by moving from one polytope to another. While many approaches identify the active set of the adjacent critical region by fixing the parameter and solving the resulting QP, it was shown that the active set of the adjacent region can in fact be inferred *a priori* (Tøndel et al., 2003):

Definition 5 (*Facet-to-Facet Property Tøndel et al., 2003*)**.** Let CR_1 and CR_2 be two full-dimensional critical regions with $\mathrm{int}\,(CR_1) \cap \mathrm{int}\,(CR_2) = \emptyset$. Then the facet-to-facet property is said to hold if $F = CR_1 \cap CR_2$ is a facet of both CR_1 and CR_2.

Theorem 6 (*Active Set of Adjacent Region Tøndel et al., 2003*)**.** *Consider an active set $k = \{i_1, i_2, \ldots, i_k\}$ and its corresponding critical region CR_0 in minimal representation, i.e. with all redundant*

[1] Other approaches for the solution of problem (2) involve graphical derivatives (Patrinos & Sarimveis, 2010) or vertex enumeration (Mönnigmann & Jost, 2012).