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Brief paper

Infinite horizon linear-quadratic Stackelberg games for discrete-time stochastic systems☆

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ABSTRACT

In this paper, we consider infinite horizon linear-quadratic Stackelberg games for a discrete-time stochastic system with multiple decision makers. Necessary conditions for the existence of the Stackelberg strategy set are derived in terms of the solvability of cross-coupled stochastic algebraic equations (CSAEs). As an important application, the hierarchical H_∞-constraint control problem for the infinite horizon discrete-time stochastic system with multiple channel inputs is solved using the Stackelberg game approach. Computational methods for solving the CSAEs are also discussed. A numerical example is provided to demonstrate the usefulness of the proposed algorithms.

1. Introduction

Many researchers have extensively investigated the dynamic games of continuous- and discrete-time systems (see Basar & Olsder, 1999 and the references therein). Recently, due to the growth of interest in multi-agent and cooperative systems, optimal cooperation and team collaboration have been widely investigated. For example, neural networks were utilized to find suitable approximations of the optimal-value function and saddle-point solutions (Zhang, Qin, Jiang, & Luo, 2014). The conditions for the existence of Pareto optimal solutions for linear-quadratic infinite horizon cooperative differential games were derived (Reddya & Engwerda, 2013). The design method for the synchronization control of discrete-time multi-agent systems on directed communication graphs has been discussed (Movric, You, Lewis, & Xie, 2013). The open-loop Nash games for a class of polynomial system via a state-dependent Riccati equation have been studied (Lizarraga, Basin, Rodriguez, & Rodriguez, 2015). Among various dynamic games, the Stackelberg game is one that involves hierarchical decisions among different decision makers, yielding significant and useful work (Abou-Kandil, 1990; Basar, Bensoussan, & Sethi, 2010; Bensoussan, Chen, & Sethi, 2013, 2014, 2015; Jungers & Oara, 2010; Jungers, Trelat, & Abou-Kandil, 2008; Jungers, Trelat, & Abou-kandil, 2011; Li, Cruz, & Simaan, 2002; Medanic, 1978; Xu, Zhang, & Chai, 2015). Although many researchers have studied Stackelberg games, they have only focused on those for finite horizon deterministic continuous- or discrete-time systems. To the best of our knowledge, the Stackelberg game for infinite horizon discrete-time systems is still unsolved. The infinite horizon Stackelberg game is difficult to solve, because it involves some higher-order algebraic nonlinear matrix equations rather than difference equations in the finite horizon case (Abou-Kandil, 1990) and the team-optimal state feedback Stackelberg strategy case (Li et al., 2002).

Recent advances in the theory of discrete-time stochastic systems have led to the reconsideration of various control problems for infinite horizon discrete-time stochastic systems (Huang, Zhang, & Zhang, 2008; Zhang, Huang, & Xie, 2008). It has been shown that the optimal or mixed H_2/H_∞ feedback controllers can be constructed by solving certain stochastic algebraic Riccati equations (SAREs). However, corresponding results have not been found in dynamic game settings, especially in dynamic games with multiple decision makers, except for Mukaidani, Tanabata, and Matsumoto (2014) who only considered Nash games. Taking into

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consideration the fact that the dynamic game approaches have found many applications in the control field, the investigation of hierarchical dynamic games for discrete-time stochastic systems is extremely promising.

The Stackelberg games for a class of stochastic systems have been considered (Mukaidani, 2013, 2014a,b; Mukaidani & Xu, 2015). In Mukaidani (2013) and Mukaidani and Xu (2015), the Stackelberg games for the continuous-time stochastic systems with multiple decision makers have been studied. In Mukaidani (2014a,b), the Stackelberg games for a discrete-time stochastic system with one leader and one follower have been studied. In particular, in Mukaidani (2013, 2014a), the mixed H_2/H_∞ control problem was investigated where the follower is interpreted as a deterministic disturbance.

In this paper, we investigated the infinite horizon linear-quadratic Stackelberg game for a class of discrete-time linear stochastic systems with state-dependent noise. The results in Mukaidani (2014a) were merely preliminary results to study a standard Stackelberg game with one follower. In this paper, we have extended the results in Mukaidani (2014a) to the Stackelberg game with multiple followers. Moreover, we have studied the hierarchical H_∞-constraint control problem with multiple decision-makers using this Stackelberg game approach. Although the Lagrange-multiplier technique was used to derive the necessary conditions in the same way as in Mukaidani and Xu (2015), the derivation procedures, and consequently the results, were completely different.

Notation: The notations used in this paper are fairly standard. The superscript T denotes matrix transpose. I_n denotes the $n \times n$ identity matrix. $\mathbb{E}[\cdot]$ denotes the expectation operator. **Tr** denotes the trace of a matrix. **vec** denotes the column vector of a matrix. δ_{ij} denotes the Kronecker delta. **block diag** denotes the block diagonal matrix. $\boldsymbol{X}$ denotes a set of $\{X_0,\ X_1, \ldots, X_N\}$. $\rho(\cdot)$ denotes the spectral radius function.

2. Preliminary

Before investigating the Stackelberg game, some useful lemmas are introduced. Consider the following discrete-time stochastic system.

$$x(k+1) = Ax(k) + Bu(k) + A_p x(k) w(k),$$
$$x(0) = x_0, \tag{1a}$$
$$y(k) = Cx(k), \tag{1b}$$

where $x(k) \in \mathbb{R}^n$ represents the state vector, $u(k) \in \mathbb{R}^m$ represents the control input, $y(k) \in \mathbb{R}^p$ represents the measured output, and $w(k) \in \mathbb{R}$ is a one-dimensional sequence of real random process defined in a complete probability space, which is a wide sense stationary, second-order process with $\mathbb{E}[w(k)] = 0$ and $\mathbb{E}[w(s)w(k)] = \delta_{sk}$ (Huang et al., 2008; Zhang et al., 2008). $l_w^2(\mathbb{N},\ \mathbb{R}^n)$ denotes the set of nonanticipative square summable stochastic processes $y = \{y(k) : y(k) \in \mathbb{R}^n\} = \{y(0),\ y(1), \ldots\}$, such that $y(k)$ is $\mathcal{F}_{k-1}$ measurable, where $\mathcal{F}_k$ denotes the σ-algebra generated by $w(s)$, $s = 0,\ 1, \ldots, k$. The l^2-norm of $y(k) \in l_w^2(\mathbb{N},\ \mathbb{R}^n)$ is defined by $\|y(k)\|^2_{l_w^2(\mathbb{N},\ \mathbb{R}^n)} := \sum_{k=0}^{\infty} \mathbb{E}[\|y(k)\|^2]$ (Huang et al., 2008; Zhang et al., 2008).

The following definition has been introduced in Huang et al. (2008) and Zhang et al. (2008).

Definition 1. An autonomous discrete-time stochastic system $x(k+1) = Ax(k) + A_p x(k) w(k)$, ($u(k) \equiv 0$ in (1a)) is called asymptotically mean square stable (*AMSS*) if for any x_0, the corresponding state satisfies $\lim_{k\to\infty} \mathbb{E}[\|x(k)\|^2] = 0$. Moreover, $(A,\ A_p\ /\ C)$ is exactly observable if $y(k) \equiv 0$ almost surely $\forall k \in \mathbb{N} \Rightarrow\ x(0) = 0$.

When system (1) is AMSS, $(A,\ A_p)$ is called stable for short. The following lemma has been proved (Huang et al., 2008).

Lemma 1. *If $(A,\ A_p)$ is stable, then for any C, the following stochastic algebraic Lyapunov equation* (SALE)

$$-P + A^T PA + A_p^T PA_p + C^T C = 0 \tag{2}$$

admits a unique solution $P \geq 0$. If $(A,\ A_p\ /\ C)$ is exactly observable, then $(A,\ A_p)$ is stable if and only if (2) has a unique solution $P > 0$. Moreover,

$$\sum_{k=0}^{\infty} \mathbb{E}\Big[x^T(k)C^T Cx(k)\Big] = \mathbb{E}\Big[x^T(0)Px(0)\Big], \tag{3}$$

where $x(k+1) = Ax(k) + A_p x(k) w(k)$.

Definition 2. System (1) is stabilizable in the mean square sense, if there exists a feedback control $u(k) = Kx(k)$ with K (a constant matrix), such that for any x_0, the closed-loop system is AMSS. The triple of matrices $(A,\ B,\ A_p)$ is called stabilizable if and only if stochastic system (1) is stabilizable.

The following lemma plays a key technical role in this paper (Huang et al., 2008; Zhang et al., 2008).

Lemma 2 (*Huang et al., 2008; Zhang et al., 2008*)**.** *The following stochastic linear-quadratic* (LQ) *control problem is considered subject to (1):*

$$\text{minimize } J(u) := \sum_{k=0}^{\infty} \mathbb{E}\Big[x^T(k)Qx(k) + u^T(k)Ru(k)\Big],$$
$$Q = Q^T \geq 0,\ R = R^T > 0. \tag{4}$$

If $(A,\ B,\ A_p)$ is stabilizable and $(A,\ A_p\ /\ \sqrt{Q})$ is exactly observable, then the following stochastic algebraic Riccati equation (SARE) *has a unique solution $P = P^*$.*

$$-P + A^T PA + A_p^T PA_p + Q$$
$$- A^T PB(R + B^T PB)^{-1} B^T PA = 0. \tag{5}$$

Furthermore, $J(u^*) = \mathbb{E}[x^T(0)P^* x(0)]$ and the optimal feedback control is given by

$$u^*(k) = K^* x(k) = -(R + B^T PB)^{-1} B^T PAx(k). \tag{6}$$

3. Stackelberg game with multiple followers

The following stochastic systems with $N+1$-decision makers involving state-dependent noise is considered.

$$x(k+1) = Ax(k) + B_0 u_0(k) + \sum_{i=1}^{N} B_i u_i(k)$$
$$+ A_p x(k) w(k), \qquad x(0) = x_0, \tag{7}$$

where $u_i(k) \in \mathbb{R}^{m_i}$, $i = 0,\ 1, \ldots, N$ represents the ith control input. Moreover, $i = 0$ represents the leader's control input, and the other values for i represent the follower's inputs. The initial state $x(0) = x_0$ is assumed to be a random variable with a covariance matrix $\mathbb{E}[x(0)x^T(0)] = I_n$.

Without loss of generality, the following basic assumption is made:

Assumption 1. $(A,\ B_i,\ A_p)$, $i = 0,\ 1, \ldots, N$ is stabilizable.