Automatica 76 (2017) 336–344

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

The maximal contractively invariant ellipsoids for discrete-time linear systems under saturated linear feedback[☆]

Yuanlong Li [a], Zongli Lin [b]

[a] Department of Automation, Shanghai Jiao Tong University, and Key Laboratory of System Control and Information Processing of Ministry of Education, Shanghai 200240, China

[b] Charles L. Brown Department of Electrical and Computer Engineering, University of Virginia, P.O. Box 400743, Charlottesville, VA 22904-4743, USA

ARTICLE INFO

Article history:
Received 21 June 2014
Received in revised form
6 June 2016
Accepted 13 September 2016

Keywords:
Actuator saturation
Ellipsoidal invariant set
Domain of attraction

ABSTRACT

In this paper, we consider the problem of determining the maximal contractively invariant ellipsoids for discrete-time linear systems with multiple inputs under saturated linear feedback. We propose an algebraic computational approach to determining such maximal contractively invariant ellipsoids. We divide the state space into several regions according to the saturation status of each input, and compute the possible maximal contractively invariant ellipsoids on each region except the region where none of inputs saturate and on their intersections. The minimal one among these possible maximal contractively invariant ellipsoids is the maximal contractively invariant ellipsoids of the system. Simulation results demonstrate the effectiveness of our methods.

1. Introduction

For a linear system asymptotically null controllable by bounded controls, except for very special cases, saturated linear feedback can only achieves local asymptotical stabilization (Sussmann & Yang, 1991). A linear system is said to be asymptotically null controllable with bounded controls if it is stabilizable in the usual linear systems theory sense and all its open loop poles are in the closed left-half plane. For such a system under saturated linear feedback, its domain of attraction in general cannot be analytically characterized. This raises an important problem of how to estimate its domain of attraction. Much effort has been made on this problem (Alamo, Cepeda, & Limon, 2005; Alamo, Cepeda, Limon, & Camacho, 2006a,b; Dai, Hu, Teel, & Zaccarian, 2009; Gomes da Silva & Tarbouriech, 1999, 2001; Hu & Lin, 2002a, 2005; Milani, 2002; Pittet, Tarbouriech, & Burgat, 1997; Tarbouriech, Garcia, Gomes da Silva, & Queinnec, 2011). For example, the authors of Alamo et al. (2006a) used a polyhedral SNS invariant set, which embeds the characteristics of saturation functions, as an estimate of the domain of attraction of discrete-time saturated linear systems. In Gomes da Silva and Tarbouriech (1999), a necessary and sufficient condition to determine the contractivity of polyhedral regions has been presented for linear discrete-time systems with saturating controls. Moreover, a Lyapunov function is composed from a group of quadratic Lyapunov functions (Hu & Lin, 2005). The level set of this composite Lyapunov function, which is the convex hull of the ellipsoids corresponding to the level sets of the individual quadratic Lyapunov functions, leads to a larger estimate of the domain of attraction (Hu & Lin, 2005).

As one of the most commonly used invariant sets, the ellipsoid, as the level set of the quadratic Lyapunov function, has been widely used in estimating the domain of attraction of saturated linear systems (Alamo et al., 2005; Gomes da Silva & Tarbouriech, 2001; Hu & Lin, 2001, 2002a; Tarbouriech et al., 2011). Based on the convex hull representation of saturated linear feedback, an optimization problem with a set of LMI constraints has been formulated to obtain a large contractively invariant ellipsoid (Alamo et al., 2005; Hu & Lin, 2002a). Since these constraints are only sufficient conditions for an ellipsoid to be contractively invariant, the resulting optimal ellipsoid is generally not the maximal contractively invariant ellipsoid. However, for both continuous-time and discrete-time linear systems with a single input under saturated linear feedback, it is proven in Hu and Lin (2002b, 2005) that these sufficient LMI conditions are also necessary, and hence the optimal ellipsoid resulting from

☆ Work supported in part by the National Natural Science Foundation of China under Grant Nos. 61221003 and 61273105, and in part by the China Postdoctoral Science Foundation under Grant Nos. 2015M580332 and 2016T90373. The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Emilia Fridman under the direction of Editor Ian R. Petersen.

E-mail addresses: liyuanlong0301@163.com (Y. Li), zl5y@virginia.edu (Z. Lin).

http://dx.doi.org/10.1016/j.automatica.2016.10.007

the optimization problem is actually the maximal contractively invariant ellipsoid. With multiple inputs under saturated linear feedback, the optimal ellipsoid can be the maximal contractively invariant ellipsoid only when certain additional conditions are satisfied (Hu & Lin, 2003). For general continuous-time linear systems with multiple inputs under saturated linear feedback, an algebraic computational approach was developed for determining the maximal contractively invariant ellipsoid (Li & Lin, 2015). For general discrete-time linear systems under saturated linear feedbacks, a set of necessary and sufficient conditions were presented in Fiacchini, Prieur, and Tarbouriech (2013) that characterize the invariance and contractivity of convex sets.

In this paper, we carry out the characterization of the maximal contractively invariant ellipsoid associated with a given positive definite matrix for discrete-time linear systems with m inputs under saturated linear feedback. An algebraic computational approach to determining such maximal contractively invariant ellipsoids is developed as follows. We first divide the state space into several regions according to the saturation status of each input. We then compute the possible maximal contractively invariant ellipsoid on each region except the one where none of inputs saturate, and then determine the minimal one among these possible maximal contractively invariant ellipsoids whose extreme states reside in the corresponding regions. Next, we compute the possible maximal contractively invariant ellipsoids on intersections between some regions, where there are k inputs that simultaneously critically saturate, $k = 1, 2, \ldots, m-1$. The smallest one of these ellipsoids determined above is then the maximal contractively invariant ellipsoid for discrete-time linear systems with multiple inputs under saturated linear feedback.

To apply this algebraic computational method, we need to solve certain polynomial equations or compute the eigenvalues of certain matrices. Software tools exist that could easily carry out such computation, for example, the polynomial toolbox of Matlab. However, the amount of computation associated with this algebraic computational method increases exponentially with the dimensions of the state and input. Note that the algebraic computational method applies to any linear system with multiple inputs under saturated linear feedbacks.

Control based on discrete-time saturated system models is extensively adopted in the practice systems subject to actuator saturation. Hence, it is necessary to analyze stability of the discrete-time saturated systems including the maximal contractively invariant ellipsoid. The algebraic computational approach aforementioned to determine the maximal contractively invariant ellipsoid is a counterpart to our recent result in Li and Lin (2015). The algebraic computation in this paper is however not a direct generalization of that in Li and Lin (2015), as the computational details in the discrete-time setting are significantly different from those in continuous-time setting.

The remainder of the paper is organized as follows. In the preliminaries section, we recall some relevant results on ellipsoidal invariant sets of discrete-time linear systems under saturated linear feedback. In Section 3, we propose an algebraic computational method to determine the maximal contractively invariant ellipsoid for discrete-time linear systems with multiple inputs under saturated linear feedback. In Section 4, numerical examples are given to demonstrate the effectiveness of the results in this paper. Section 5 concludes the paper.

We will use standard notation. For a vector $u = [u_1\ u_2 \ldots u_m]^{\mathrm{T}}$, $|u|_\infty := \max_i |u_i|$. For two integers $k_1, k_2, k_1 < k_2$, $I[k_1, k_2] := \{k_1, k_1+1, \ldots, k_2\}$. For a positive definite $P \in \mathbf{R}^{n\times n}$ and a positive scalar ρ, $\mathcal{E}(P, \rho) := \{x \in \mathbf{R}^n : x^{\mathrm{T}}Px \le \rho\}$. For a set $\mathcal{S}$, we use $\mathcal{S}^\circ$ and $\partial\mathcal{S}$ to denote its interior and its boundary. For a matrix $H \in \mathbf{R}^{m\times n}$, $\mathcal{L}(H) := \{x \in \mathbf{R}^n : |Hx|_\infty \le 1\}$. For a matrix A, $\mathrm{He}(A) = A^{\mathrm{T}} + A$. I_n stands for an n-dimensional identity matrix.

2. Preliminaries

Consider the following discrete-time system

$$x^+ = Ax + B\mathrm{sat}(Fx), \tag{1}$$

where $x \in \mathbf{R}^n$ denotes the state vector, x^+ is the successor state, $F \in \mathbf{R}^{m\times n}$ is the feedback gain, and $\mathrm{sat} : \mathbf{R}^m \to \mathbf{R}^m$ denotes the vector valued standard saturation function, which is defined as $\mathrm{sat}(u) = [\mathrm{sat}(u_1), \mathrm{sat}(u_2), \ldots, \mathrm{sat}(u_m)]^{\mathrm{T}}$, $\mathrm{sat}(u_i) = \mathrm{sgn}(u_i)\min\{1, |u_i|\}$. A signal u_i is said to saturate if $|u_i| > 1$ and it is said to saturate critically if $|u_i| = 1$. Given a positive definitive matrix $P \in \mathbf{R}^{n\times n}$, let $V(x) = x^{\mathrm{T}}Px$. The ellipsoid $\mathcal{E}(P, \rho)$ is said to be contractively invariant if

$$\begin{aligned}\Delta V(x) &= V(x^+) - V(x)\\ &= (Ax + B\mathrm{sat}(Fx))^{\mathrm{T}}P(Ax + B\mathrm{sat}(Fx)) - x^{\mathrm{T}}Px\\ &< 0, \quad \forall x \in \mathcal{E}(P, \rho) \setminus \{0\}.\end{aligned}$$

In this paper, we assume that, for the given matrix P, $(A + BF)^{\mathrm{T}}P(A + BF) - P < 0$. This assumption is necessary since it guarantees the existence of a contractively invariant ellipsoid for system (1). We further assume system (1) is local stable, and hence the maximal contractively invariant ellipsoid exists. The following fact is clear.

Fact 1. Let

$$\rho_{\mathrm{c}} := \sup\{\rho > 0 : \ \mathcal{E}(P, \rho) \text{ is contractively invariant}\}.$$

Then, a $\rho^* > 0$ is such that $\rho^* = \rho_{\mathrm{c}}$ if and only if $\Delta V(x) < 0$, $\forall\, x \in \mathcal{E}^\circ(P, \rho^*) \setminus \{0\}$, and $\Delta V(x_0) = 0$ for some $x_0 \in \partial\mathcal{E}(P, \rho^*)$.

For later use in this paper, we denote the maximal contractively invariant ellipsoid as $\mathcal{E}(P, \rho_{\mathrm{c}})$, and refer to an $x_0 \in \partial\mathcal{E}(P, \rho_{\mathrm{c}})$ such that $\Delta V(x_0) = 0$ as the extreme state.

We next recall the convex hull representation of a saturated linear feedback from Hu and Lin (2001). Let $\mathcal{D}$ denote the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. There are 2^m such matrices in $\mathcal{D}$, and we label them as D_i, $i \in I[1, 2^m]$. Denote $D_i^- = I - D_i$. Clearly, $D_i^- \in \mathcal{D}$.

Lemma 2 (Hu & Lin, 2001). *Let $F, H \in \mathbf{R}^{m\times n}$. Then, for any $x \in \mathcal{L}(H)$,*

$$\mathrm{sat}(Fx) \in \mathrm{co}\left\{D_iFx + D_i^-Hx : i \in I[1, 2^m]\right\},$$

where co stands for the convex hull.

By Lemma 2, the m-dimensional nonlinear function $\mathrm{sat}(Fx)$ is expressed as a linear combination of the 2^m auxiliary linear feedbacks. With this expression, the conditions under which the ellipsoid $\mathcal{E}(P, \rho)$ is a contractively invariant set of the closed system (1) were established in Hu and Lin (2001) as follows.

Theorem 3. *Given an ellipsoid $\mathcal{E}(P, \rho)$. If there exists an $H \in \mathbf{R}^{m\times n}$ such that*

$$\begin{aligned}&(A + B(D_iF + D_i^-H))^{\mathrm{T}}P(A + B(D_iF + D_i^-H))\\ &\quad - P < 0, \quad \forall\, i \in I[1, 2^m],\end{aligned}$$

and $\mathcal{E}(P, \rho) \subseteq \mathcal{L}(H)$, then $\mathcal{E}(P, \rho)$ is contractively invariant for system (1).

Theorem 3 provides a set of sufficient conditions under which $\mathcal{E}(P, \rho)$ is contractively invariant. Moreover, as established in Hu and Lin (2005), for single input systems, that is, $m = 1$, the resulting linear matrix inequalities that characterize the invariance of an ellipsoid $\mathcal{E}(P, \rho)$ are also necessary.

Next, we will focus on the largest invariant ellipsoid $\mathcal{E}(P, \rho)$ that satisfies the conditions of Theorem 3 for a given positive