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Technical communique

Controllability to the origin implies state-feedback stabilizability for discrete-time nonlinear systems☆

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ABSTRACT

The problem of finite-time state-feedback stabilizability of discrete-time nonlinear systems has been considered in this technical communique. Two assertions have been proved. First, if the system is N-step controllable to the origin, then there is a state feedback control law for which the trajectory of the closed-loop system converges to the origin in N steps. Second, if the system is asymptotically controllable to the origin and satisfies the controllability rank condition at the origin, then there is a state feedback control law for which the trajectory of the closed-loop system converges to the origin in finite steps.

1. Introduction

From control theoretic point of view, one of the most important properties of a system is stability. Controllability assures the existence of an open-loop control law, but in many cases, a state-feedback control law is preferable.

For continuous-time systems, the relation between asymptotic controllability and state-feedback stabilizability has been established in Clarke, Ledyaev, Sontag, and Subbotin (1997). In Clarke et al. (1997), it has been shown that there is a discontinuous state feedback stabilizing control law for a system which is asymptotically controllable. Because the discontinuity of the control law arises naturally in stabilization and optimization problems, discontinuous control laws have been studied in many papers, e.g. Ceragioli (2002), Clarke (2011), Clarke, Ledyaev, Rifford, and Stern (2000), and Rifford (2002). Regarding the time-scale, the stability property is classified into two categories: asymptotic property and finite-time property, and recently, there has been a growing interest on the latter property (Bhat & Bernstein, 2000; Haimo, 1986; Huang, Lin, & Yang, 2005; Moulay & Perruquetti, 2006).

For discrete-time systems, problems related to controllability have been extensively studied in Albertini and Sontag (1994), Jakubczyk and Sontag (1990) and Sontag and Wirth (1998). However, these works do not deal with state-feedback stabilization problem. State feedback stabilization problem of discrete-time nonlinear systems has been studied for past decades (e.g. Byrnes, Lin, & Ghosh, 1993; Jiang, Lin, & Wang, 2004; Jiang & Wang, 2001; Kazakos & Tsinias, 1994; Ornelas-Tellez, Sanchez, Loukianov, & Rico, 2014), but researches dealing with nonsmooth or discontinuous control laws are relatively rare (Meadows, Henson, Eaton, & Rawlings, 1995; Simões, Nijmeijer, Tsinias, & Sontag, 1996). The connection between controllability and stabilizability has been analyzed in an early work by Sontag for piecewise linear systems (Sontag, 1981). In Sontag (1981), both finite-time stability and asymptotic stability have been analyzed, but the scope is limited to piecewise linear systems. More general systems have been dealt with in Kellet and Teel (2004) and Kreisselmeier and Birkhölzer (1994), but they concentrate on asymptotic properties. To the best of the author's knowledge, the connection between controllability to the origin and finite-time state-feedback stabilizability has not been investigated for systems more general than piecewise linear systems. The objective of this technical communique is to fill the gap.

In the following, we prove two facts. First, if a discrete-time nonlinear system is N-step controllable to the origin (precise definition of this notion is given below), then there is a (possibly

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discontinuous) state feedback control law for which the trajectory of the closed-loop system converges to the origin in N steps. Second, if the system is asymptotically controllable to the origin and satisfies the controllability rank condition at the origin (again, precise definitions of these notions are given below), then there is a (possibly discontinuous) state feedback control law for which the trajectory of the closed-loop system converges to the origin in finite steps (the required steps may differ for different initial conditions). Our construction explicitly uses the axiom of choice.

A preliminary version of this manuscript is available in arxiv.org (Hanba, 2015).

2. Main results

Consider a discrete-time time-invariant nonlinear system of the form

$$x(t+1) = f(x(t), u(t)), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $t \in \mathbb{Z}_{\geq 0}$ is the time, and $\mathbb{Z}_{\geq 0}$ denotes the set of nonnegative integers. It is assumed that $(x, u) = (0, 0)$ is an equilibrium of (1), that is, $f(0, 0) = 0$. Henceforth, we use the following notations: $u[t_0, t_1]$ denotes the finite sequence of inputs $(u(t_0), \ldots, u(t_1))$, and $u[t_0, \infty)$ denotes the infinite sequence of inputs $(u(t_0), u(t_0 + 1), \ldots)$. We identify $u[t_i, t_i]$ with $u(t_i)$, and $u[t_i, t_j]$ with the empty sequence if $t_j < t_i$. For $t \geq t_0$, $\phi(t, t_0, x_0, u)$ denotes the trajectory of (1) initialized at $t = t_0$ by x_0 and driven by the input (with $\phi(t_0, t_0, x_0, u) = x_0$), and it is also interpreted as the composition of functions defined recursively: $\phi(t, t_0, x_0, u) = f(\phi(t-1, t_0, x_0, u), u(t-1))$. In the subsequent analysis, we sometimes fix t at some N and regard $\phi(N, t_0, x_0, u)$ as a function of x_0 and $u[t_0, N-1]$. The notation $\phi(t, t_0, x_0, u)$ implicitly assumes that the input is at least defined over the time interval $[t_0, t-1]$. Because we are dealing with a time-invariant system, the value of the initial time t_0 is immaterial. Hence, without loss of generality, we assume that $t_0 = 0$.

Properties related to controllability have been used by many researchers for different meanings. The following two definitions are analogous to the one employed in Sontag (1981).

Definition 1. The system (1) is said to be N-step controllability to the origin if $\exists N > 0, \forall x_0 \in \mathbb{R}^n, \exists u[0, N-1], \phi(N, 0, x_0; u) = 0$.

Definition 2. The system (1) is said to be asymptotically controllability to the origin if $\forall x_0 \in \mathbb{R}^n, \exists u[0, \infty), \phi(t, 0, x_0; u)$ converges to the origin as $t \to \infty$.

Another popular notion on controllability is the generalization of controllability rank condition of linear systems to nonlinear systems given below.

Definition 3. The system (1) is said to be rank controllable at the origin if $\exists N > 0$,

$$\operatorname{rank} \frac{\partial \phi(N, 0, x_0, u)}{\partial u[0, N-1]} = n \tag{2}$$

on an open neighborhood of $(x, u(0), \ldots, u(N-1)) = (0, 0, \ldots, 0)$, where $\partial\phi(N, 0, x_0, u)/\partial u[0, N-1]$ denotes the partial derivative of ϕ with respect to the variable $u[0, \ldots, N-1]$ with the re-interpretation that $u[0, \ldots, N-1]$ is the vector $(u^T(0), u^T(1), \ldots, u^T(N-1))^T$, and $\cdot^T$ denotes the transpose of a vector.

In Sontag and Wirth (1998), the rank condition (2) does not have an explicit name, but a tuple $(x, u(0), \ldots, u(N-1))$ that satisfies (2) is called 'regular'. Note that we are specializing at $(x, u(0), \ldots, u(N-1)) = (0, 0, \ldots, 0)$.

Remark 1. Definitions 1 and 2 are conditions that are not checkable for general nonlinear systems and should blindly be assumed, except for polynomial systems with rational coefficients (Nešić & Mareels, 1998). On the other hand, for a discrete-time nonlinear system, $\phi(t, 0, x_0; u)$ is merely a composition of known functions, hence Definition 3 is checkable, although computationally demanding.

Because we are dealing with a time-invariant system, it is preferable that our controller is time-invariant as well.

Definition 4. The system (1) is said to be (globally) finite-time stabilizable by a state feedback controller if there is a control law $\upsilon(x)$ defined in $\mathbb{R}^n$ with the property that

$$\forall x_0 \in \mathbb{R}^n, \ \exists N_x > 0, \quad \phi(N_x, 0, x_0; \upsilon(x)) = 0.$$

On the other hand, if

$$\exists N > 0, \ \forall x_0 \in \mathbb{R}^n, \quad \phi(N, 0, x_0; \upsilon(x)) = 0,$$

then the system (1) is said to be (globally) N-step stabilizable by a state feedback controller.

Because we do not deal with local property in this technical communique, henceforth, we omit the term 'global'.

Our first objective is to show the following proposition, which is a straightforward extension of the result given in Sontag (1981) for piecewise linear systems.

Lemma 1. *The system (1) is N-step controllable to the origin if and only if it is N-step stabilizable by a state feedback controller.*

Proof. One direction is straightforward: if (1) N-step stabilizable by a state feedback controller, it is N-step controllable to the origin by the input determined by the controller.

For the converse, let $A_0 = \{0\}$, and inductively define $A_k = \{x \in \mathbb{R}^n : \exists u, f(x, u) \in A_{k-1}\}$. Because the system is N-step controllable to the origin, $A_N = \mathbb{R}^n$. For the sequence $(A_0, A_1, \ldots, A_N)$ and each $x \in \mathbb{R}^n$, let

$$i_x = \min\{i : x \in A_i\}. \tag{3}$$

If $i_x = 0$, let $\upsilon(x) = 0$ (recall that $A_0 = \{0\}$). Otherwise, the set $U_x = \{u : f(x, u) \in A_{i_x - 1}\}$ is nonempty. Pick an element $u_x \in U_x$ (here, we use the axiom of choice), and let $\upsilon(x) = u_x$. Because i_x is uniquely determined for each x, $\upsilon(x)$ is well defined. Let $(x_t)_{t \in \mathbb{Z}_{\geq 0}}$ be the trajectory of (1) initialized with x_0 and driven by the control law $\upsilon(x)$, that is, $x_t = \phi(t, 0, x_0; \upsilon(x))$. We prove that $i_{x_t} = 0$ for some $t \leq N$ by contradiction. Suppose that $\forall t, i_{x_t} > 0$. Then, $x_0 \in A_{i_{x_0}}$ for some $i_{x_0} \leq N$ and hence $x_1 = f(x_0, \upsilon(x_0)) \in A_{i_{x_0} - 1}$, and by (3), $i_{x_1} \leq i_{x_0} - 1$. Inductively, assume that $x_j \in A_{i_{x_j}}$ with $i_{x_j} \leq i_{x_0} - j$. Then, $x_{j+1} = f(x_j, \upsilon(x_j)) \in A_{i_{x_j} - 1}$, hence $i_{x_{j+1}} \leq i_{x_0} - (j+1)$. Therefore, for any j, $i_{x_j} \leq i_{x_0} - j$. But this is impossible, because $0 \leq i_{x_0} \leq N$ and $0 \leq i_{x_j} \leq N$. □

Remark 2. In Kellet and Teel (2004), the following fact is shown (Theorem 15 of Kellet and Teel (2004)) (in this manuscript, the input space $\mathcal{U}$ and the target set $\mathcal{A}$ of Kellet and Teel (2004) are $\mathbb{R}^m$ and $\{0\}$, respectively, hence we omit them in quoting the result of Kellet and Teel (2004)).

Let $\sigma : [0, \infty) \to [0, \infty)$ be a nondecreasing function, $\overline{B}_u = \{u \in \mathbb{R}^m : \|u\| \leq 1\}$, and define the set-valued map $F(x)$ by $F(x) = f(x, \sigma(\|x\|)\overline{B}_u)$. If $F(x)$ is continuous in the sense of Kellet and Teel (2004), $F(x)$ is a non-empty compact set for each x, and (1) is uniformly globally asymptotically controllable to the origin with σ controls in the sense of Kellet and Teel (2004), then there is a feedback function such that the origin is robustly globally asymptotically stable.