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Stability analysis of systems with time-varying delays via the second-order Bessel–Legendre inequality[☆]

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ABSTRACT

This paper is concerned with the delay-dependent stability of systems with time-varying delays. The novelty relies on the use of the second-order Bessel–Legendre integral inequality which is less conservative than the Jensen and Wirtinger-based inequalities. *Unlike similar contributions, the features of this inequality are fully integrated into the construction of augmented Lyapunov–Krasovskii functionals* leading to novel stability criteria expressed in terms of linear matrix inequalities. The stability condition is tested on some classical numerical examples illustrating the efficiency of the proposed method.

1. Introduction

Over the last two decades, the delay-dependent stability analysis of systems with time-varying delays via Lyapunov–Krasovskii functional (LKF) method has received much attention (see e.g., Fridman, 2014; Xu, Lam, Zhang, & Zou, 2015 and the references therein). The crucial technical steps regarding this method are related to both the selection of the functional and the use of accurate bounding methods to derive linear matrix inequalities (LMIs).

Among the bounding methods, the Jensen inequality (Gu, Kharitonov, & Chen, 2003) has been widely adopted, although at the price of an unavoidable conservatism. Recently, much attention has been paid to reducing this conservatism. A recent direction of research consists in deriving extended-like Jensen inequalities, which encompasses Jensen inequality through the introduction of additional quadratic terms. The first result in this direction led to the so-called Wirtinger-based inequality developed in Seuret and Gouaisbaut (2013) and Seuret, Gouaisbaut, and Fridman (2013). Then, further extensions were derived by Zeng et al. using a free-matrix-based integral inequality (Zeng, He, Wu, & She, 2015a). More recently, generalized integral inequalities were developed in Seuret and Gouaisbaut (2014, 2015) based on Bessel inequality and Legendre polynomials, which include Jensen and Wirtinger-based inequalities and also the recent inequalities based on auxiliary functions-based inequality (Park, Lee, & Lee, 2015; Zeng, He, Wu, & She, 2015b) as particular cases. The main interest of such inequalities is that the conservatism can be reduced arbitrarily. These new inequalities have been mainly employed to the case of *constant* discrete or *distributed* delays (see Seuret & Gouaisbaut, 2015; Zeng et al., 2015b). The first attempt to derive stability condition for time-varying discrete delay was proposed in Park et al. (2015). In this paper, we pursue the method provided in Park et al. (2015) and derive more accurate conditions.

In this paper, we develop novel stability criteria for linear systems with time-varying delays using the particular case of the second-order Bessel–Legendre inequality (i.e. the Bessel–Legendre inequality from Seuret and Gouaisbaut (2015) with $N = 2$, where N is the degree of Legendre polynomials) of the integral inequalities proposed in Seuret and Gouaisbaut (2014, 2015), that recovers the inequality provided in Park et al. (2015) and Zeng et al. (2015b). The main contributions are as follows:

1. The features of the limited Bessel–Legendre inequality are fully integrated into the construction of the LKFs.
2. Less conservative stability criteria are derived in terms of LMIs although the computational complexity is higher.

Numerical examples taken from the literature illustrate the efficiency of our results. In particular, the numerical results for the constant delay case coincide with the ones achieved in

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Zeng et al. (2015b) and are very close to the analytical bounds of constant delays preserving the stability.

Throughout the paper, in addition to usual notations, the set $\mathbb{S}_+^n$ denotes the set of symmetric positive definite matrices. For any matrices A, B, $\text{diag}(A, B)$ stands for the matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Moreover, for any square matrix A, we define $\text{He}(A) = A + A^T$.

2. Problem formulations

2.1. System data

Consider a linear system with time-varying delays:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_1 x(t - h(t)), & t \geq 0, \\ x(t) = \phi(t), & -h_2 \leq t \leq 0, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $A, A_1 \in \mathbb{R}^{n\times n}$ are constant matrices, and ϕ is the initial condition. The time-varying delay $h(t)$ is continuous and satisfies

$$0 \leq h_1 \leq h(t) \leq h_2, \qquad h_{12} \triangleq h_2 - h_1. \tag{2}$$

There is no restriction on the derivative of the delay function.

2.2. Limited Bessel–Legendre inequality

Let us first recall the inequality that will be the core of the paper. It corresponds to the inequality recently shown in Zeng et al. (2015b), which is also a particular case of the Bessel–Legendre inequality of Seuret and Gouaisbaut (2015). The proof of this inequality can be found in Zeng et al. (2015b) or in Seuret and Gouaisbaut (2015).

Lemma 1. *For a given matrix $R \in \mathbb{S}_+^n$, any differentiable function x in $[a, b] \to \mathbb{R}^n$, the inequality*

$$\int_a^b \dot{x}^T(u) R \dot{x}(u) \mathrm{d}u \geq \frac{1}{b-a} \Omega^T \text{diag}(R, 3R, 5R) \Omega \tag{3}$$

holds, where

$$\Omega = \begin{bmatrix} x(b) - x(a) \\ x(b) + x(a) - \dfrac{2}{b-a}\displaystyle\int_a^b x(u)\mathrm{d}u \\ x(b) - x(a) - \dfrac{6}{b-a}\displaystyle\int_a^b \delta_{a,b}(u) x(u) \mathrm{d}u \end{bmatrix},$$
$$\delta_{a,b}(u) = 2\left(\frac{u-a}{b-a}\right) - 1.$$

Remark 1. The inequality (3) encompasses the Wirtinger-based inequality of Seuret and Gouaisbaut (2013) with the help of the third component of the vector Ω. This improvement requires the introduction of an extra signal $\int_a^b \delta_{a,b}(u)x(u)\mathrm{d}u$ in addition to $\int_a^b x(u)\mathrm{d}u$, $x(b)$ and $x(a)$.

2.3. Parameter-dependent matrix inequalities

The following lemma is for an alternative formulation of the reciprocally convex combination inequality from Park, Ko, and Jeong (2011).

Lemma 2. *For any given matrix $R \in \mathbb{S}_+^n$, assume that there exists a matrix $X \in \mathbb{R}^{n\times n}$ such that $\begin{bmatrix} R & X \\ X^T & R \end{bmatrix} \succeq 0$. Then, the following inequality holds*

$$\begin{bmatrix} \dfrac{1}{\alpha}R & 0 \\ 0 & \dfrac{1}{1-\alpha}R \end{bmatrix} \succeq \begin{bmatrix} R & X \\ X^T & R \end{bmatrix}, \quad \forall \alpha \in (0,\ 1).$$

Alternatively, we present another lemma, which refers to the classical bounding technique (Moon, Park, Kwon, & Lee, 2001).

Lemma 3. *For any matrices $R_1 \in \mathbb{S}_+^n, R_2 \in \mathbb{S}_+^n, Y_1 \in \mathbb{R}^{2n\times n}$ and $Y_2 \in \mathbb{R}^{2n\times n}$, the following inequality holds*

$$\begin{bmatrix} \dfrac{1}{\alpha}R_1 & 0 \\ 0 & \dfrac{1}{1-\alpha}R_2 \end{bmatrix} \succeq \Theta_M(\alpha), \quad \forall \alpha \in (0,\ 1),$$

where

$$\begin{aligned} \Theta_M(\alpha) = {} & \text{He}\left(Y_1 \begin{bmatrix} I_n & 0_{n\times n} \end{bmatrix} + Y_2 \begin{bmatrix} 0_{n\times n} & I_n \end{bmatrix}\right) \\ & - \alpha Y_1 R_1^{-1} Y_1^T - (1-\alpha) Y_2 R_2^{-1} Y_2^T. \end{aligned}$$

The notable difference between Lemmas 2 and 3 is that, in Lemma 3, the lower bound depends explicitly on the uncertain parameter α. This dependence on α eventually leads to a reduction of conservatism at the price of additional decision variables.

3. Stability analysis of time-varying delay systems

In this section, based on Lemma 1 together with Lemma 2 or 3, two novel stability criteria are provided for system (1) with time-varying delays. For the simplicity of presentation, we will use in this section the following notations:

$$\begin{aligned} e_i &= [0_{n\times(i-1)n}\ I_n\ 0_{n\times(14-i)n}],\ i = 1, \ldots, 14, \\ G_2 &= [e_1^T - e_2^T\ \ e_1^T + e_2^T - 2e_5^T\ \ e_1^T - e_2^T - 6e_6^T]^T, \\ G_3 &= [e_2^T - e_3^T\ \ e_2^T + e_3^T - 2e_7^T\ \ e_2^T - e_3^T - 6e_8^T]^T, \\ G_4 &= [e_3^T - e_4^T\ \ e_3^T + e_4^T - 2e_9^T\ \ e_3^T - e_4^T - 6e_{10}^T]^T, \\ \Gamma &= [G_3^T\ \ G_4^T]^T, \qquad \Sigma = Ae_1 + A_1 e_3. \end{aligned} \tag{4}$$

and

$$\begin{aligned} \eta_0(t) &= [x^T(t)\ x^T(t-h_1)\ x^T(t-h(t))\ x^T(t-h_2)]^T, \\ \eta_1(t) &= \frac{1}{h_1}\left[\int_{-h_1}^0 x_t^T(s)\mathrm{d}s \quad \int_{-h_1}^0 \delta_1(s) x_t^T(s)\mathrm{d}s\right]^T, \\ \eta_2(t) &= \frac{1}{h(t)-h_1}\left[\int_{-h(t)}^{-h_1} x_t^T(s)\mathrm{d}s \quad \int_{-h(t)}^{-h_1} \delta_2(s) x_t^T(s)\mathrm{d}s\right]^T, \\ \eta_3(t) &= \frac{1}{h_2-h(t)}\left[\int_{-h_2}^{-h(t)} x_t^T(s)\mathrm{d}s \quad \int_{-h_2}^{-h(t)} \delta_3(s) x_t^T(s)\mathrm{d}s\right]^T, \\ \eta_4(t) &= (h(t)-h_1)\eta_2(t), \quad \eta_5(t) = (h_2 - h(t))\eta_3(t). \\ \eta_6(t) &= \left[\int_{-h_2}^{-h_1} x_t^T(s)\mathrm{d}s \quad h_{12}\int_{-h_2}^{-h_1} \delta_4(s) x_t^T(s)\mathrm{d}s\right]^T, \end{aligned} \tag{5}$$

and where the functions δ_i, for $i = 1, \ldots, 4$, which refer to the functions $\delta_{a,b}$ given in Lemma 1, are given by

$$\begin{aligned} \delta_1(s) &= 2\frac{s+h_1}{h_1} - 1, & \delta_2(s) &= 2\frac{s+h(t)}{h(t)-h_1} - 1, \\ \delta_3(s) &= 2\frac{s+h_2}{h_2-h(t)} - 1, & \delta_4(s) &= 2\frac{s+h_2}{h_{12}} - 1. \end{aligned}$$