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Symmetrizing quantum dynamics beyond gossip-type algorithms[☆]

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ABSTRACT

Recently, consensus-type problems have been formulated in the quantum domain. Obtaining average quantum consensus consists in the dynamical symmetrization of a multipartite quantum system while preserving the expectation of a given global observable. In this paper, two improved ways of obtaining consensus via dissipative engineering are introduced, which employ on quasi local preparation of mixtures of symmetric pure states, and show better performance in terms of purity dynamics with respect to existing algorithms. In addition, the first method can be used in combination with simple control resources in order to engineer pure Dicke states, while the second method guarantees a stronger type of consensus, namely single-measurement consensus. This implies that outcomes of local measurements on different subsystems are perfectly correlated when consensus is achieved. Both dynamics can be randomized and are suitable for feedback implementation.

1. Introduction

In quantum as well as in classical systems, symmetry is a fundamental concept, and a key tool in investigating dynamical systems. In particular, suitable dynamical symmetries, or equivalently the existence of preserved quantities, prevent controllability for quantum dynamics (Altafini & Ticozzi, 2012; D'Alessandro, 2007; Dirr, Helmke, Kurniawan, & Schulte-Herbrüggen, 2009), while they allow for the existence of protected sets of states (Knill & Laflamme, 1997; Knill, Laflamme, & Viola, 2000; Viola, Knill, & Lloyd, 1999; Zanardi, 1999). Symmetric quantum states and subspaces have a key role in the description of quantum systems obeying Bose–Einstein statistics (Kardar, 2007), they are related to thermalization (Eisert, Friesdorf, & Gogolin, 0000), and have a key role in quantum information (Nielsen & Chuang, 2002). Prototypical states used to illustrate and exploit intrinsically quantum correlations between information units, or *entanglement between qubits* in the quantum information jargon, are the maximally entangled states named after Greenberg–Horn–Zeilinger (GHZ) (Greenberger, Horne, & Zeilinger, 1989) and the W states (Dür, Vidal, & Cirac, 2000): both are invariant with respect to permutation of their subsystems.

In Mazzarella, Ticozzi, and Sarlette (2015) it is shown that a class of classical dynamics obtaining asymptotic consensus can be seen as symmetrizing dynamics, leading to permutation-invariant states. The same underlying idea led to dynamics for symmetrizations in the quantum realm (Mazzarella, Sarlette, & Ticozzi, 2015). There, it is shown how quantum consensus algorithms can be used in combination with simple local controls and measurements in order to prepare pure states and estimate the size of a network. This type of symmetrizing dynamics and their convergence properties, as well as their continuous-time counterpart, have been further studied in Shi, Dong, Petersen, and Johansson (2015), Shi, Fu, and Petersen (2015) and Ticozzi, Mazzarella, and Sarlette (2014). All the symmetrizing dynamics that have been proposed so far, however, are based on combinations of permutation operators, and hence they share two common properties: (1) they are *unital*, i.e. the maximally mixed state is preserved; (2) they attain symmetric state consensus, which is effectively consensus on the *statistical* properties of the variables of interest, but there is no algorithm that can attain actual consensus on the output of each local measurement. Due to the contraction properties of the considered maps, unital dynamics entail that the purity of the quantum state cannot be augmented by the consensus-reaching dynamics. With a degradation of purity, the fragile quantum correlations encoded in the state are typically lost, and stronger notions of consensus are out of reach. If the consensus-achieving dynamics improve purity, they can be

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instrumental to distributed and robust preparation of interesting states, like Dicke (including W) or GHZ states (Ticozzi & Viola, 2012, 2014), as we shall demonstrate in the following.

In the effort of overcoming these issues, two types of symmetrizing dynamics are proposed, which allow for asymptotic symmetrization of the state of a multipartite quantum system with respect to the permutation group acting on the subsystems. Both dynamics are composed of quasi-local maps and are robust with respect to randomization, and are hence suitable for distributed and unsupervised implementations—as are their classical and classically-inspired counterparts.

The first one attains an asymptotically symmetric state while preserving the expectation value of a global observable, reaching *average symmetric state consensus* in the language of Mazzarella, Sarlette et al. (2015). It does so by selecting and preparing a pure symmetric state (a *Dicke* state Dicke, 1954; Garraway, 2011) in each eigen-subspace of the observable of interest. It is also shown how these dynamics are instrumental to preparation of globally entangled pure states. The proposed protocol uses only single system operations and pairwise interactions.

The second dynamics are the first proposal of a quasi-local protocol that obtains a stronger type of quantum consensus, called single-measurement consensus (Mazzarella, Sarlette et al., 2015). This type of consensus, which implies the symmetry of the state but for which the latter is not sufficient, is the closest in spirit to classical consensus: after single measurement consensus is reached, the measurement of a local observable quantity on any subsystem will force the whole network of systems to "agree" on the result, i.e. yielding perfectly correlating results. Notice that, in contrast with classical consensus, the consensus value may not be determined before it is actually measured.

Both methods rely on essentially quantum features of the system and, while obtaining a symmetric pure state that has a specified average of an observable is not viable in general, they allow for a final purity that is typically better than the one offered by gossip-type algorithms. Both are suitable for implementation via discrete-time feedback (Bolognani & Ticozzi, 2010), and can be combined with local initialization procedures in order to actually *prepare perfectly pure and entangled symmetric states*. This is explicitly shown for the first proposed algorithm (see Corollary 2). These dynamics can be seen as the discrete-time equivalent of conditional preparation of entangled states, in the spirit of Ticozzi and Viola (2014).

The paper structure is as follows: in Section 2 a brief review of the relevant quantum consensus definitions and the ideas underlying existing consensus-achieving dynamics is provided; Section 3 begins by explaining why trying to obtain quantum consensus with just pure states is impossible, and continues by providing the form and the convergence proofs for two novel symmetrizing dynamics. While these can be straightforwardly extended to networks of d-dimensional systems, as those considered in the introductory section the presentation here focuses on qubit networks and pairwise interactions. Qubit systems are easier to visualize and yet relevant for applications, some of which have been discussed in Mazzarella, Sarlette et al. (2015). Pairwise interactions are the minimal that can be allowed for interacting dynamics, and if a more forgiving locality constraint is in place, a set of effective pairwise interactions will also be allowed under this locality notion. Section 4 illustrates the behavior of the two dynamics, and compares them with the existing consensus algorithm.

2. Background

2.1. Quantum consensus states

Consider a multipartite system composed of m isomorphic subsystems, labeled with indices $i = 1, \ldots, m$, with associated Hilbert space $\mathcal{H}^m := \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_m \simeq \mathcal{H}^{\otimes m}$, with $\dim(\mathcal{H}_i) = \dim(\mathcal{H}) = n$ and $2 \leqslant n < \infty$. This multipartite system will act as our *quantum network*. We shall use Dirac's notation: $|\psi\rangle$ denote vectors of $\mathcal{H}$, $\langle\psi|$ denote their dual linear functional. $\mathfrak{B}(\mathcal{H})$ denotes the set of linear operators on $\mathcal{H}$, which in our finite-dimensional setting are in a one-to-one relationship with complex matrices. States are associated to density operators, namely linear, trace-one, positive-semidefinite operators on $\mathcal{H}$, with their set denoted by $\mathfrak{D}(\mathcal{H}) \subset \mathfrak{B}(\mathcal{H})$. Observable quantities can be associated to Hermitian operators on $\mathcal{H}$, denoted by $\mathfrak{H}(\mathcal{H})$. The *support* of an Hermitian operator is the orthogonal complement to its kernel. Given a state ρ and an observable X, the expectation of X according to ρ is computed as $\mathrm{Tr}(\rho X)$.

For any operator $X \in \mathfrak{B}(\mathcal{H})$, denote by $X^{\otimes m}$ the tensor product $X \otimes X \otimes \cdots \otimes X$ with m factors. Given an operator $\sigma \in \mathfrak{B}(\mathcal{H})$, denote by $\sigma^{(i)}$ the local operator:

$$\sigma^{(i)} := I^{\otimes(i-1)} \otimes \sigma \otimes I^{\otimes(m-i)}.$$

Permutations of quantum subsystems are expressed by a unitary operator $U_\pi \in \mathfrak{U}(\mathcal{H})$, which is uniquely defined by

$$U_\pi(X_1 \otimes \cdots \otimes X_m)U_\pi^\dagger = X_{\pi(1)} \otimes \cdots \otimes X_{\pi(m)}$$

for any operators $X_1, \ldots, X_m$ in $\mathfrak{B}(\mathcal{H})$, where π is a permutation of the first m integers. $\mathfrak{P}$ is the set of such π. A state or observable is said to be *permutation invariant* if it commutes with all the subsystem permutations.

In Mazzarella, Sarlette et al. (2015) a number of potential extensions of the idea of classical consensus to a quantum network were proposed, and their merit discussed in depth. The ones relevant to this work are:

Definition 1 (*SSC*)**.** A state $\rho \in \mathfrak{D}(\mathcal{H}^m)$ is in *Symmetric State Consensus* (SSC) if, for each unitary permutation U_π,

$$U_\pi\, \rho\, U_\pi^\dagger = \rho.$$

Definition 2 (σSMC)**.** Given $\sigma \in \mathfrak{B}(\mathcal{H})$ with spectral decomposition $\sigma = \sum_{j=1}^d s_j \Pi_j \in \mathfrak{H}(\mathcal{H})$, a state $\rho \in \mathfrak{D}(\mathcal{H}^m)$ is in *Single σ-Measurement Consensus* (σSMC) if:

$$\mathrm{Tr}(\Pi_j^{(k)}\Pi_j^{(\ell)}\rho) = \mathrm{Tr}(\Pi_j^{(\ell)}\rho), \tag{1}$$

for all $k, \ell \in \{1, \ldots, m\}$, and for each j.

The definition of σSMC is the unique, among those proposed, that requires that the outcomes of σ measurements on different subsystems be exactly the same *for each trial*. Consider the set of projections $\{\Pi_j\}_{j=1}^d$ as in Definition 2, and let us define

$$\Pi_{\mathrm{SMC}} = \sum_{j=1}^{d} \Pi_j^{\otimes m}.$$

It has been shown that a state is in σSMC if and only if it holds

$$\mathrm{Tr}(\Pi_{\mathrm{SMC}}\rho) = 1, \tag{2}$$

or equivalently

$$\Pi_{\mathrm{SMC}}\rho\Pi_{\mathrm{SMC}} = \Pi_{\mathrm{SMC}}\rho = \rho. \tag{3}$$

Furthermore σSMC for σ with non-degenerate spectrum implies SSC, while the converse implications do not hold. Lastly, it is impossible for a state to be σSMC with respect to all $\sigma \in \mathfrak{H}(\mathcal{H})$: this means that σSMC cannot be strengthened to a single-measurement equivalent of SSC. The proofs of these statements are given in Mazzarella, Sarlette et al. (2015).

It is worth remarking how all these definitions could be given for classical systems, in the context of consensus for random variables or for probability distributions of the state values. In