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Optimal portfolios with maximum Value-at-Risk constraint under a hidden Markovian regime-switching model[☆]

Dong-Mei Zhu[a], Yue Xie[b,1], Wai-Ki Ching[c], Tak-Kuen Siu[d]

[a] School of Economics and Management, Southeast University, Nanjing, China

[b] College of Economics and Management, Zhejiang University of Technology, Zhejiang, China

[c] Advanced Modeling and Applied Computing, Department of Mathematics, The University of Hong Kong, Hong Kong

[d] Department of Applied Finance and Actuarial Studies, Faculty of Business and Economics, Macquarie University, Sydney, Australia

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ABSTRACT

This paper studies an optimal portfolio selection problem in the presence of the Maximum Value-at-Risk (MVaR) constraint in a hidden Markovian regime-switching environment. The price dynamics of n risky assets are governed by a hidden Markovian regime-switching model with a hidden Markov chain whose states represent the states of an economy. We formulate the problem as a constrained utility maximization problem over a finite time horizon and then reduce it to solving a Hamilton–Jacobi–Bellman (HJB) equation using the separation principle. The MVaR constraint for n risky assets plus one riskless asset is derived and the method of Lagrange multiplier is used to deal with the constraint. A numerical algorithm is then adopted to solve the HJB equation. Numerical results are provided to demonstrate the implementation of the algorithm.

1. Introduction

In modern finance, optimal portfolio allocation of different assets is a prominent issue. Markowitz (1952) pioneered the use of a mean–variance approach in formulating optimal allocation problems. His approach reduces the problem to the situation that one only needs to maximize the expected return under an acceptable level of risk in a single period. The risk level is measured by the variance of the return. Then, Merton (1969, 1971) extended this single-period model to a continuous-time framework which reflects the market environment better. Closed form solutions were then derived using stochastic optimal control techniques with the premise that the coefficients in the price process of the risky assets are constant. However, this assumption may not be realistic. Therefore, some researchers started investigating asset allocation models with non-constant parameters. For example, Boyle and Yang (1997) employed a multi-factor stochastic interest rate model from Duffie and Kan (1996) for the bond price dynamics in their asset allocation model. Lim and Zhou (2002) analyzed a continuous time mean–variance portfolio selection problem with random market coefficients.

Recently, Markovian regime-switching models have been widely applied in economics and finance, since it can give a reasonably good description for some important stylized features of the price dynamics of assets. The applications of Markov-switching time series models to economics and econometrics were introduced by Hamilton (1989). The use of Markovian regime-switching models for portfolio selection has received much attention. For example, in Zhou and Yin (2003), the state of the market model which would affect the parameters in the stock price process was described by Markovian regime switching models with observable regimes. The efficient portfolios were derived explicitly in closed forms for their Markowitz mean–variance portfolio selection model using techniques of stochastic linear–quadratic control. In Elliott, Siu, and Badescu (2010), they modeled the evolution of the state of the economy by a hidden Markov chain model and assumed that the "true" state of the underlying economy is unobservable. An explicit solution was derived in their mean–variance portfolio selection model using the

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E-mail addresses: dongmeizhu86@gmail.com (D.-M. Zhu), xieyue1988moon@163.com (Y. Xie), wching@hku.hk (W.-K. Ching), Ken.Siu@mq.edu.au, ktksiu2005@gmail.com (T.-K. Siu).

1 Fax: +852 2559 2225.

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stochastic maximum principle. Honda (2003) studied the optimal portfolio choice when the mean returns of a risky asset depend on a hidden Markov chain. Some other works on optimal asset allocation in hidden Markovian regime-switching economy are, for example, Baeuerle and Rieder (2007), Elliott and Siu (2012), Korn, Siu, and Zhang (2011), Sass and Haussmann (2004), Shen and Siu (2015) and Siu (2011, 2012, 2013, 2015, 2016), amongst others.

Value-at-Risk (VaR) is one of the popular risk measures used in market risk management. Informally speaking, VaR describes the maximum expected loss during a given period at a given level of confidence. VaR has been used as a risk constraint in portfolio optimization. For example, Basak and Shapiro (2001) considered the optimal portfolio policies when VaR is imposed as a constraint though they pointed out that the use of the VaR constraint may lead to sub-optimal results. Yiu (2004) derived the VaR constraint for multiple risky assets and a riskless asset, and found that investments in risky assets are reduced when the VaR constraint becomes active. Typically, VaR is derived under the assumption that the parameters, such as interest rate, drift and volatility, in the price dynamics of assets are assumed to be known beforehand so that some standard distributions such as a normal distribution may be applied to compute VaR. However, if these parameters depend on the state of the underlying economy which may switch over time, then the values of these parameters may be uncertain or unknown. The Maximum Value-at-Risk (MVaR) may provide a conservative way to describe risks under this situation. It is defined as the maximum value of the VaRs of the portfolio at different states of the underlying economy in a given time. Yiu, Liu, Siu, and Ching (2010) discussed a utility maximization problem constrained by the MVaR. In their paper, the state of the economy is modeled by an observable Markov chain. While it seems that there is a relatively little work on optimal portfolio allocation in multiple risky assets in a hidden Markovian regime-switching economy using the MVaR as a risk constraint.

In this paper we extend the portfolio allocation model with one risky asset in Yiu et al. (2010) to a more general situation of multiple risky assets. Furthermore, the state of an economy was assumed to be observable in Yiu et al. (2010). However, in practice, the state of the underlying economy may be unobservable or not directly observable. It is assumed that the hidden state of the economy is described by a continuous-time hidden Markov chain. Similar method can be found in Elliott and van der Hoek (1997). The price dynamics of n risky assets follow an n-dimensional Geometric Brownian Motion (GBM) where the value of the drift is supposed to switch over time according to the states of the underlying hidden Markov chain. The MVaR is derived for n risky assets plus one riskless asset and imposed as a constraint. We formulate this optimal portfolio allocation problem as a constrained utility maximization problem. Then using the separation principle in Elliott et al. (2010), the problem can be separated into two problems: a filtering-estimation problem and a constrained stochastic control problem. A robust form of filtering equations is presented to estimate the unknown parameters by applying the gauge transformation technique which is proposed by Clark (1978) and applied in Elliott, Malcolm, and Tsoi (2003); Elliott et al. (2010). In this way, solving the constrained stochastic control problem is converted to solving a Hamilton–Jacobi–Bellman (HJB) equation, where the MVaR constraint is handled by the method of Lagrange multiplier. A numerical algorithm is used to solve the HJB equation for the optimal constrained portfolio numerically.

The rest of this paper is structured as follows. In Section 2, the price dynamics of n risky assets are presented. The optimal portfolio selection problem without constraint is formulated as a maximization of the expected utility over a given period. Section 3 discusses the separation principle. The corresponding MVaR is also derived to describe investment risks. In Section 4, the filters for the hidden states of the economy are presented. In Sections 5 and 6, a numerical algorithm and numerical results are presented. Finally, concluding remarks are given in Section 7.

2. Model dynamics and portfolio allocation problem

In this section, we consider a continuous-time economy with a finite time horizon $\mathcal{T} := [0, T]$. All of the uncertainties are described by a complete probability space $(\Omega, \mathcal{F}, P)$, where P is a real-world probability measure. Let $\mathbf{y}'$ denote the transpose of a matrix or a vector $\mathbf{y}$ and $\mathbb{1}_{m\times n}$ denote an $m \times n$-dimensional matrix whose entries are all equal to one. The model dynamics described are those in standard hidden Markovian regime-switching financial models. Similar models have been used for portfolio selection in the literature (see, for example, Elliott & Siu, 2012; Elliott et al., 2010; Korn et al., 2011; Sass & Haussmann, 2004; Shen & Siu, 2015; Siu, 2011, 2012, 2013, 2015, 2016, and the relevant literature therein).

Let $X := \{X(t)\}_{t\le T}$ be a continuous-time, finite-state Markov chain with state space $\varepsilon := \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$, where $\mathbf{e}_i$ is the unit vector in $\mathcal{R}^N$ with one in the ith position and zero elsewhere. This convention of the state space of the chain was adopted in, for example, Elliott, Aggoun, and Moore (1995). Then as in Elliott et al. (1995), a semi-martingale representation for the chain is given as follows:

$$X(t) = X(0) + \int_0^t AX(u)du + M(t),$$

where $A := [a_{ij}]_{N\times N}$ is a time invariant rate matrix of the chain and $\{M(t)|t \in \mathcal{T}\}$ is a martingale under P. The element a_{ij} in A is the instantaneous intensity of the transition of the chain X from State $\mathbf{e}_j$ to State $\mathbf{e}_i$. Here the states of the chain X are interpreted as hidden states of an economy.

We consider an optimal portfolio allocation problem with n risky assets and one riskless asset. Suppose the price process of the riskless asset which is denoted as $S_0 := \{S_0(t)|t \in \mathcal{T}\}$ follows:

$$S_0(t) = \exp(rt) \quad \text{and} \quad S_0(0) = 1.$$

Here the interest rate r is assumed to be a positive constant. The price process of the n risky assets, denoted by $\mathbf{S} = \{\mathbf{S}(t)|t \in \mathcal{T}\}$, satisfies:

$$d\mathbf{S}(t) = D(\mathbf{S}(t))\mu(t)dt + D(\mathbf{S}(t))\boldsymbol{\sigma} dW(t) \quad \text{and} \quad \mathbf{S}(0) = s_0.$$

Here $\{W(t)|t \in \mathcal{T}\}$ is an n-dimensional standard Brownian motion and $D(\mathbf{S}(t))$ is the diagonal matrix of the vector $\mathbf{S}(t) = (S_1(t), \ldots, S_n(t))'$. The volatility $\boldsymbol{\sigma} = (\sigma_{ij})_{n\times n}$ is a constant non-singular matrix. Readers interested in stochastic volatility models can refer to Pham and Quenez (2001). Shen and Siu (2013) investigated the pricing of variance swaps under stochastic interest rate and volatility. They incorporated the stochastic interest rate process and separated it from the volatility process using techniques of forward measure changes. The situation when r and σ are time-varying or stochastic processes will lead to complication in the estimation of the model. In this case, the standard EM algorithm may not work well and the statistical properties such as asymptotic properties of the estimators may not be available. The drift $\mu(t)$ is assumed to depend on the state of the economy and is given as follows:

$$\mu(t) = \boldsymbol{\mu} X(t),$$

where $\boldsymbol{\mu} = (\mu_{ij})$ is an $n \times N$ matrix.

Both the drift process and the Brownian motion are assumed to be unobservable to the investor. The only observable information is the price process $\mathbf{S}$. The importance of incorporating the model uncertainty of the drift is discussed in, for example, Elliott and Siu (2012) and Elliott et al. (2010). Instead of considering the price process directly, we consider the log return process $Y := \{Y(t)|t \in \mathcal{T}\}$. Here $Y(t) = (Y_1(t), Y_2(t), \ldots, Y_n(t))'$ and $Y_i(t) = \ln(S_i(t)/S_i(0))$. It is well-known that by Itô's lemma,

$$dY(t) = g(t)dt + \boldsymbol{\sigma} dW(t),$$