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Greedy controllability of finite dimensional linear systems☆

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ABSTRACT

We analyse the problem of controllability for parameter dependent linear finite-dimensional systems. The goal is to identify the most distinguished realisations of those parameters so to better describe or approximate the whole range of controls. We adapt recent results on greedy and weak greedy algorithms for parameter dependent PDEs or, more generally, abstract equations in Banach spaces. Our results lead to optimal approximation procedures that, in particular, perform better than simply sampling the parameter-space to compute the controls for each of the parameter values. We apply these results for the approximate control of finite-difference approximations of the heat and the wave equation. The numerical experiments confirm the efficiency of the methods and show that the number of weak-greedy samplings that are required is particularly low when dealing with heat-like equations, because of the intrinsic dissipativity that the model introduces for high frequencies.

1. Introduction and problem formulation

We analyse the problem of controllability for linear finite-dimensional systems submitted to parametrised perturbations, depending on unknown parameters in a deterministic manner.

In previous works we have analysed the property of averaged control looking for a control, independent of the values of these parameters, designed to perform well, in an averaged sense (Lazar & Zuazua, 2014; Zuazua, 2014).

Here we analyse the complementary issue of determining the most relevant values of the unknown parameters so to provide the best possible approximation of the set of parameter dependent controls. Our analysis is based on previous work on reduced modelling and (weak) greedy algorithms for parameter dependent PDEs and abstract equations in Banach spaces (Buffa, Maday, Patera, Prudhomme, & Turinici, 2012; Cohen & DeVore, 2016), which we adapt to the present context.

The problem is relevant in applications, as in practice the models under consideration are often not completely determined, submitted to unknown or uncertain parameters, either of deterministic or of stochastic nature. It is therefore essential to develop robust analytical and computational methods, not only allowing to control a given model, but also to deal with parameter-dependent families of systems in a stable and computationally efficient way.

Both reduced modelling and the control theory have experienced successful real-life implementations (we refer to the book Lery et al., 2011 for a series of such interactions with the European industry). The merge of this two theories will allow variety of applications in all fields involving problems modelled by parameter dependent systems (fluid dynamics, aeronautics, meteorology, economics, etc.).

Although the greedy control is applicable to more general control problems and systems, here we concentrate on controllability issues and, to better illustrate the main ideas of the new approach, we focus on linear finite-dimensional systems of parameter dependent ODEs. Infinite-dimensional systems, as a first attempt to later consider PDE models, are discussed separately in Section 6, as well as in the Conclusion section.

Consider the finite dimensional linear control system

$$\begin{cases} \dfrac{d}{dt}x(t,\nu) = \mathbf{A}(\nu)x(t,\nu) + \mathbf{B}(\nu)u(t,\nu), \quad 0 < t < T, \\ x(0) = x^0. \end{cases} \tag{1}$$

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In (1) the (column) vector valued function $x(t,\nu) = \big(x_1(t,\nu), \dots, x_N(t,\nu)\big) \in \mathbf{R}^N$ is the state of the system at time t governed by dynamics determined by the parameter $\nu \in \mathcal{N} \subset \mathbf{R}^d$, $d \geq 1$, $\mathcal{N}$ being a compact set, $\mathbf{A}(\nu)$ is a $N \times N$-matrix governing its free dynamics and $u = u(t,\nu)$ is a M-component control vector in $\mathbf{R}^M$, $M \leq N$, entering and acting on the system through the control operator $\mathbf{B}(\nu)$, a $N \times M$ parameter dependent matrix. In the sequel, to simplify the notation, d/dt will be simply denoted by $'$.

The matrices $\mathbf{A}$ and $\mathbf{B}$ are assumed to be Lipschitz continuous with respect to the parameter ν. However, some of our analytical results (Section 6) will additionally require analytic dependence conditions on ν.

Here, to simplify the presentation, we have assumed the initial datum $x^0 \in \mathbf{R}^N$ to be controlled, to be independent of the parameter ν. Despite of this, the matrices $\mathbf{A}$ and $\mathbf{B}$ being ν-dependent, both the control and the solution will depend on ν. Similar arguments allow to handle the case when x^0 also depends on the parameter ν, which will be discussed separately.

We address the controllability of this system whose initial datum x^0 is given, known and fully determined. We assume that the system under consideration is controllable for all values of ν. This can be ensured to hold, for instance, assuming that the controllability condition is satisfied for some specific realisation ν_0 and that the variations of $\mathbf{A}(\nu)$ and $\mathbf{B}(\nu)$ with respect to ν are small enough.

In these circumstances, for each value of ν there is a control of minimal $[L^2(0,T)]^M$-norm, $u(t,\nu)$. This defines a map, $\nu \in \mathcal{N} \to [L^2(0,T)]^M$, whose regularity is determined by that of the matrices entering in the system, $\mathbf{A}(\nu)$ and $\mathbf{B}(\nu)$.

Here we are interested in the problem of determining the optimal selection of a finite number of realisations of the parameter ν so that all controls, for all possible values of ν, are optimally approximated.

More precisely, the problem can be formulated as follows.

Problem 1. Given a control time $T > 0$ and arbitrary initial data x^0 and final target $x^1 \in \mathbf{R}^N$, we consider the set of controls of minimal $[L^2(0,T)]^M$-norm, $u(t,\mathcal{N})$, corresponding to all possible values $\nu \in \mathcal{N}$ of the parameter satisfying the controllability condition:

$$x(T,\nu) = x^1. \tag{2}$$

This set of controls is compact in $[L^2(0,T)]^M$.

Given $\varepsilon > 0$ we aim at determining a family of parameters $\nu_1, \dots, \nu_n$ in $\mathcal{N}$, whose cardinal n depends on ε, so that the corresponding controls, denoted by $u_1, \dots, u_n$, are such that for every $\nu \in \mathcal{N}$ there exists $u_\nu^\star \in \operatorname{span}\{u_1, \dots, u_n\}$ steering the system (1) in time T within the ε distance from the target x^1, i.e. such that

$$\|x(T,\nu) - x^1\| \leq \varepsilon. \tag{3}$$

Here and in the sequel, in order to simplify the notation, we denote by u_ν the control $u(t,\nu)$, and similarly we use the simplified notation $\mathbf{A}_\nu, \mathbf{B}_\nu, x_\nu$.

Note that, in practice, the controllability condition (2) is relaxed to the approximate one (3). This is so since, in practical applications, when performing numerical approximations, one is interested in achieving the targets within a given error. This fact is also intrinsic to the methods we employ and develop in this paper, and that can only yield optimal procedures to compute approximations of the exact control, which turn out to be approximate controls in the sense of (3).

This problem is motivated by the practical issue of avoiding the construction of a control function u_ν for each new parameter value ν which, for large systems, although theoretically feasible by the uniform controllability assumption, would be computationally expensive. By the contrary, the methods we develop try to exploit the advantages that a suitable choice of the most representative values of ν provides when computing rapidly the approximation of the control for any other value of ν, ensuring that the system is steered to the target within the given error (3).

Of course, the compactness of the parameter set $\mathcal{N}$ and the Lipschitz-dependence assumption with respect to ν make the goal to be feasible. It would suffice, for instance, to apply a *naive* approach, by taking a fine enough uniform mesh on $\mathcal{N}$ to achieve the goal. However, our aim is to minimise the number of spanning controls n and to derive the most efficient approximation. The *naive* approach is not suitable in this respect.

To achieve this goal we adapt to the present frame of finite-dimensional control, the theory developed in recent years based on greedy and weak-greedy algorithms for parameter dependent PDEs or abstract equations in Banach spaces, which optimise the dimension of the approximating space, as well as the number of steps required for its construction.

The rest of this paper is organised as follows. In Section 2 we summarise the needed controllability results for finite-dimensional systems and reformulate Problem 1 in terms of the corresponding Gramian operator. Section 3 is devoted to the review of (weak) greedy algorithms, while their application to the control problem under consideration and its solution is provided in the subsequent section.

The computational cost of the greedy control approach is analysed in Section 5. Section 6 contains a generalisation of the approach to infinite dimensional problems followed by a convergence analysis of the greedy approximation errors with respect to the dimension of the approximating space. Section 7 contains numerical examples and experiments for finite-difference discretisations of 1-D wave and heat problems. The paper is closed pointing towards future development lines of the greedy control approach.

2. Preliminaries on finite dimensional control systems. Problem reformulation

In order to develop the analysis in this paper it is necessary to derive a convenient characterisation of the control of minimal norm u_ν, as a function of the parameter ν. This can be done in a straightforward manner in terms of the Gramian operator. In this section we briefly summarise the most basic material on finite-dimensional systems that will be used along this article (we refer to Micu & Zuazua, 2005 and Zuazua, 2006 for more details).

Consider the finite-dimensional system of dimension N:

$$x' = \mathbf{A}x + \mathbf{B}u, \quad 0 \leq t \leq T;\; x(0) = x^0, \tag{4}$$

where x is the N-dimensional state and u is the M-dimensional control, with $M \leq N$.

This corresponds to a specific realisation of the system above for a given choice of the parameter ν. We omit however the presence of ν from the notation since we are now considering a generic linear finite-dimensional system.

Here $\mathbf{A}$ is an $N \times N$ matrix with constant real coefficients and $\mathbf{B}$ is an $N \times M$ matrix. The matrix $\mathbf{A}$ determines the dynamics of the system and the matrix $\mathbf{B}$ models the way M controls act on it.

In practice, it is desirable to control the N components of the system with a low number of controls, the best possible case being the one of scalar controls: $M = 1$.

Recall that system (4) is said to be *controllable* when every initial datum $x^0 \in \mathbf{R}^N$ can be driven to any final datum x^1 in $\mathbf{R}^N$ in time T. This controllability property can be characterised by a necessary and sufficient condition, which is of purely algebraic nature, the so called *Kalman condition*: System (4) is controllable if and only if

$$\operatorname{rank}[\mathbf{B}, \mathbf{AB}, \dots, \mathbf{A}^{N-1}\mathbf{B}] = N. \tag{5}$$