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Brief paper

On reachable sets for positive linear systems under constrained exogenous inputs ☆

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ABSTRACT

This paper focuses on positive linear time-invariant systems with constant coefficients and specific exogenous disturbance. The problem of finding a hyper-pyramid to bound the set of the states that are reachable from the origin in the Euclidean space is addressed, subject to inputs whose $(1,\ 1)$-norm or $(\infty,\ 1)$-norm is bounded by a prescribed constant. The Lyapunov approach is applied and a bounding hyper-pyramid is obtained by solving a set of inequalities. Iterative procedures (with an adjustable parameter) for reducing the hyper-volume of the bounding hyper-pyramid for the reachable set are proposed.

1. Introduction

When an admissible control signal of a linear dynamic system is constrained in some way, the transfer of the system state from the origin to an arbitrary terminal state is generally not possible. Under some input constraints, the collection of all possible states to which the system can be transferred from the origin is referred to as the reachable set. The bounding of reachable states was first considered for linear systems in the late 1960s in the context of state estimation and it has later received a lot of attention in parameter estimation (see Durieu, Walter, & Polyak, 2001 and references therein). For linear systems, an ellipsoidal bound of the reachable set is often used to contain all the reachable states under zero initial conditions (Chen, Cheng, Zhong, Yang, & Kang, 2015; Fridman & Shaked, 2003; Goncharova & Ovseevich, 2016; Kang & Zhong, 2015; Kim, 2008; Nam, Pathirana, & Trinh, 2015; Trinh, Nam, Pathirana, & Le, 2015; Zhang, Lam, & Xu, 2014; Zuo, Wang, Chen, & Wang, 2014). The idea may also be used for solving the peak-to-peak minimization problem (Abegor, Nagpal, & Poolla, 1996) or control problems with saturating actuators (Hu, Lin, & Chen, 2002; Tarbouriech, Garcia, & Gomes da Silva, 2002). A linear matrix inequality (LMI) solution to the reachable set bounding problem was given in Boyd, El Ghaoui, Feron, and Balakrishnan (1994) via the Lyapunov approach. However, to the best of the authors' knowledge, no related work has been devoted to the reachable set bounding problem for *positive* systems. The novelty of the present work is that we derive, for the first time, a *hyper-pyramidal* bound on the reachable set of positive linear system under various types of norm-bounded disturbances.

On the other hand, a quantitative treatment of the performance and robustness of control systems requires the introduction of appropriate signal and system norms, which measure the magnitudes of the involved signals and system operators. As discussed in Chen, Lam, Li, and Shu (2013a), some frequently used performance measures such as the H_∞ or L_2–L_∞ norms are based on the L_2 signal space, which are not very natural, in some situations, to describe the features of practical systems with positivity. On the other hand, the 1-norm of a vector-valued signal could provide a useful description for positive systems

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which measures the size of the input and/or output signals by summing the quantities of the non-negative components at a given time (Chen, Lam, Li, & Shu, 2013b), and the L_1-norm measures the accumulation of all the components over time (Xiang & Xiang, 2013), which are more appropriate, for instance, when they represent the amount of material or the number of animals in a species. Thus, the performance of positive systems can be well evaluated based on the L_1-gain (that is, the induced norm of L_1 input and L_1 output). Naturally, the linear Lyapunov function can be applied as a valid candidate for stability analysis and controller synthesis of positive systems. By using the linear Lyapunov approach, stability analysis, L_1-gain performance analysis and control design have been discussed for positive continuous-time linear systems (Briat, 2013; Haddad & Chellaboina, 2005; Wang & Huang, 2013) and positive continuous-time switched systems with delays (Haddad, Chellaboina, & Rajpurohit, 2004; Liu & Dang, 2011; Liu, Yu, & Wang, 2010). In this paper, two types of admissible input signals are considered based on 1-norm and/or ∞-norm. Under such specific classes of inputs, reachable set bounding and controller synthesis in the Euclidean space with hyper-pyramids are developed for positive linear systems.

The remaining parts of this article are organized as follows. In Section 2, preliminaries are presented for positive continuous-time systems. The input sets and the corresponding reachable sets are defined and their properties are briefly discussed in Section 3. Based on the characterizations of two classes of exogenous inputs, sufficient conditions are derived to find a hyper-pyramid that bounds the set of the states which are reachable from the origin. The problem can be tackled by finding an admissible positive vector subject to inequality constraints, and thus two iterative schemes are presented to construct a bounding hyper-pyramid for the system reachable set. The state-feedback synthesis problem is considered in Section 4 such that the closed-loop system is positive and its reachable set can be restrained within a certain hyper-pyramid.

Notations:

$\mathbb{N}_+$, $\mathbb{R}$	Set of positive integers, set of real numbers
$\mathbb{R}^n$	Set of n-dimensional real vectors
$\mathbb{R}^{m\times n}$	Set of $m \times n$ real matrices
$\bar{\mathbb{R}}^n_+$, $\mathbb{R}^n_+$	Nonnegative and positive orthants of $\mathbb{R}^n$
e_i	Vector with 1 in ith position and 0 elsewhere
$\mathbf{1}$, I	Vector $[1, 1, \ldots, 1]^T$, identity matrix
$\lambda_i(A)$	ith eigenvalue of matrix A
A^T	Transpose of A
$\|x(t)\|_1$	$\sum_{i=1}^n \lvert x_i(t)\rvert,\ x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$
$\|x(t)\|_\infty$	$\max_{i=1}^n \lvert x_i(t)\rvert,\ x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$
$\|\omega\|_{1,1}$	$\int_0^\infty \|\omega(s)\|_1 ds$ (called L_1 norm in Chen et al., 2013b)
$\|\omega\|_{\infty,1}$	ess $\sup_{t\geq 0}\|\omega(t)\|_1$

Moreover, $x \geq\geq 0$ ($x \gg 0$) denotes every component of x is nonnegative (positive) (x is called nonnegative (positive)); $A \geq\geq 0$ ($A \gg 0$) denotes every entry of matrix A is nonnegative (positive) (A is called nonnegative (positive)). A set $\mathcal{P}$ in a linear vector space is convex if $\alpha x_1 + (1-\alpha)x_2 \in \mathcal{P}$, for all $x_1, x_2 \in \mathcal{P}$ and $\alpha \in [0, 1]$. Furthermore, $\mathcal{P}$ is a convex cone if it is convex and in addition $\alpha x \in \mathcal{P}$ for all $x \in \mathcal{P}$ and all $\alpha > 0$. Vectors and matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

2. Mathematical preliminaries

Definition 1 (*Jönsson, 2001*)**.** Let $\mathbb{V}$ is a linear vector space, $s_k\colon \mathbb{V} \to \mathbb{R}$. The constraint $s_k(y) \geq 0$ is said to be regular if there exists $y^* \in \mathbb{V}$ such that $s_k(y^*) > 0,\ k = 1, 2, \ldots, N,\ N \in \mathbb{N}_+$.

Lemma 1 (*Jönsson, 2001*)**.** *Let $s_k\colon \mathbb{R}^m \to \mathbb{R}$, $s_k(y) = g_k^T y + h_k$, for $k = 0, 1, \ldots, N$, be linear functionals defined in a linear vector space $\mathbb{R}^m$, where $g_k \in \mathbb{R}^m$, $h_k \in \mathbb{R}$, and $N \in \mathbb{N}_+$. If $s_k(y)$ is regular for $k = 1, 2, \ldots, N$, the following two conditions are equivalent.*

(S_1) *$s_0(y) \geq 0$, for all $y \in \mathbb{R}^m$ such that $s_k(y) \geq 0,\ k = 1, 2, \ldots, N$.*
(S_2) *There exist scalars $\tau_k \geq 0$, $k = 1, 2, \ldots, N$ such that $s_0(y) - \sum_{k=1}^N \tau_k s_k(y) \geq 0,\ \forall y \in \mathbb{R}^m$.*

Lemma 1 is a linear version of the classical quadratic S-procedure (Fradkov, 1973; Yakubovich, 1971). It is a valid tool for verifying the non-negativity of a linear function $s_0(y)$ under a finite number of linear constraints $s_k(y) \geq 0$ ($k = 1, 2, \ldots, N$) since condition (S_2), in general, is much simpler to verify than condition (S_1).

Definition 2 (*Farina & Rinaldi, 2000*)**.** $A \in \mathbb{R}^{n\times n}$ is Metzler if its off-diagonal elements are nonnegative, that is, $A_{(i,j)} \geq 0,\ i, j = 1, 2, \ldots, n,\ i \neq j$.

3. Reachable sets with exogenous inputs

Consider positive linear dynamic systems described by the vector differential equation:

$$\dot{x}(t) = Ax(t) + B_\omega \omega(t) \tag{1}$$

where $x(t) \in \bar{\mathbb{R}}^n_+$, $\omega(t) \in \bar{\mathbb{R}}^m_+$ are the system state and an exogenous input signal, respectively, $A \in \mathbb{R}^{n\times n}$ and $B_\omega \in \bar{\mathbb{R}}^{n\times m}_+$ are constant system matrices, B_ω is nonzero.

When the input $\omega(t)$ is taken into account, we try to provide a fundamental characterization on the reachable set of system (1). The problem of bounding the reachable set of a linear system within an ellipsoid of $\mathbb{R}^n$, which has center at the origin, arises in many fields (Boyd et al., 1994). However, since the reachable states of positive system (1) with $\omega(t)$ lie in the first orthant as $x(t) \geq 0$, the set containing the reachable set is a subset of $\bar{\mathbb{R}}^n_+$. To pose such a problem precisely, we need to know how the subset is described and what bounding criterion should be used (volume, semi-major or semi-minor axis, for instance).

3.1. Estimation of reachable sets

A hyper-pyramid $\mathcal{C}_p$ of the form, for a given $p \in \mathbb{R}^n_+$:

$$\mathcal{C}_p = \{\xi \in \bar{\mathbb{R}}^n_+ \mid p^T \xi \leq 1\} \tag{2}$$

will be considered in this paper for bounding the set of reachable states of system (1) under zero initial conditions. Such a hyper-pyramid is a subset of the positive orthant and it is convex (that is, the segment connecting two points of the hyper-pyramid belongs to the hyper-pyramid itself). It is clear that all the system states $x(t)$ are contained in $\mathcal{C}_p$ if and only if $p^T x(t) \leq 1$. Next, characterization on the hyper-pyramid $\mathcal{C}_p$ will be established for two possible classes of the exogenous input signal $\omega(t)$ based on the L_1-norm and 1-norm.

Case (i): $\omega \in \Omega_{1,1} \triangleq \left\{\omega \in \bar{\mathbb{R}}^m_+ \mid \|\omega\|_{1,1} \leq 1\right\}$.

In this case, the disturbance $\omega(t)$ is considered to have L_1-norm no greater than unity.

Theorem 1. *The reachable set of positive system (A, B_ω) with zero initial conditions and input $\omega \in \Omega_{1,1}$ is bounded by the hyper-pyramid in (2) with vector $p \in \mathbb{R}^n_+$ if*

$$A^T p \leq\leq 0; \qquad B_\omega^T p \leq\leq \mathbf{1}. \tag{3}$$

Proof. Construct a Lyapunov function $V(x(t)) = p^T x(t)$ with $p \gg 0$, then $V(x(0)) = p^T x(0) = 0$. The derivative of $V(x(t))$ along the