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Brief paper

Stabilization of bilinear sparse matrix control systems using periodic inputs☆

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ABSTRACT

In this brief paper, using results from the theory of averaging of bilinear systems, we provide a graph theoretic characterization for the existence of periodic control inputs that stabilize a sparse matrix system to the origin. In particular, we introduce a class of extensions to the directed graph corresponding to a given sparse matrix system, which when contains a stable sparse matrix system implies that the original system is stabilizable using periodic inputs. When this condition holds, we provide a systematic procedure for designing such controllers. Our technical approach combines ideas from the theory of bilinear control systems and averaging theory with graph theoretic results.

1. Introduction

With the increase of interest in large-scale systems interconnected over networks and influenced by feedbacks, there has been a surge of activities within the control community for understanding the stability and controllability properties of these systems, see for example (Aguilar & Gharesifard, 2015; Langbort, Chandra, & D'Andrea, 2004; Lin, 1974; Rahmani, Ji, Mesbahi, & Egerstedt, 2009; Reinschke, 1988; Rotkowitz & Lall, 2006; Tanner, 2004). Most of the recent effort has been devoted to linear networked control systems, as they reveal valuable information about the linearization of general networked control systems. On the stabilizability front, which is the main focus of this paper, the interesting recent work (Belabbas, 2013) investigates the limitations imposed by the topology of interconnections in linear networked control systems via the notion of *(stable) sparse matrix spaces (SMSs)*.

As demonstrated in Belabbas (2013), one can associate a directed graph to a sparse matrix system, where the directed outgoing edges from a vertex show which subsystems influence this vertex's dynamics. The weights on these edges are free design parameter. It is shown that the topology of this directed graph, for example the existence of Hamiltonian subdigraphs, provides necessary/sufficient conditions for the existence of a choice for the free parameters that stabilizes this sparse matrix system to the origin. One, however, might still wonder if it is possible to use some control inputs that change the values of the free parameters on the edges over time and allow us to achieve the asymptotic stability to the origin, when such necessary conditions fail to hold. In fact, in many scenarios of complex systems, the external inputs can be coupled with the interconnections among the subsystems (Williamson, 1977). Bilinear control systems capture the simplest form of such couplings, and yet still model many practical scenarios, for example chemical and microbial cell-growth model (Williamson, 1977). Moreover, as noted recently (Ghosh & Ruths, 2014), changing the intensity of interconnections between the subsystems of a complex system can provide tools for controlling their evolutions. The controllability and stabilizability of bilinear systems are well-studied topics (Brockett, 1972a,b, 1973; d'Alessandro, Isidori, & Ruberti, 1974; Elliott, 2009; Piechottka & Frank, 1992), and yet there are still many open questions about their most basic controllability properties, see Elliott (2009), Ornik (2013) and references therein.

Having this in mind and motivated by Baillieul (1995), in this brief paper we study the stabilization of *bilinear* sparse matrix *control* systems by periodic control inputs on a few edges. The idea behind using periodic control inputs goes back to classical topics on periodic averaging in dynamical systems, where under appropriate conditions, the asymptotic stability properties of the averaged flow can reveal valuable information about the domain of attraction of the original flow (Bogoliubov & Mitropolsky, 1961; Sanders & Verhulst, 1985; Teel, Peuteman, & Aeyels, 1999). Interestingly,

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for bilinear control systems, as shown in Baillieul (1995) in the context of energy shaping of mechanical systems, periodic inputs can create new control directions. More importantly, under restrictive assumptions, when the averaged dynamics is stable, there exists periodic control inputs with high enough frequency that stabilizes the original bilinear system to the origin. One of the main objective of this brief note is to demonstrate that, in spite of the restrictions imposed in Baillieul (1995), there is a large class of sparse matrix systems which are not stable, i.e., there exists no choice of constant values for the free parameters which make it Hurwitz, but are stabilizable using periodic inputs, influencing the intensity of interconnections on only a few edges. In particular, using the matching of the original directed graph, we characterize a class of extensions to a given sparse matrix system and if one of these extensions is a stable sparse matrix system, the original system can be stabilized to the origin using periodic inputs on a selected number of edges. We also give a systematic way for finding appropriate control edges and designing the corresponding periodic inputs on them. Various examples demonstrate our results.

1.1. Organization

After introducing some mathematical preliminaries, we formulate the problem statement in Section 2. In Section 3, after recalling some notions from bilinear control and averaging, we define some graph theoretic notions along with some elementary key results that characterize the edges selected to be controlled by periodic inputs. Section 4 contains our main result on sufficient conditions for stabilization of sparse matrix systems by periodic inputs. We also introduce a procedure for designing such controls, and demonstrate our results by various examples. Section 5 gathers our conclusions and ideas for future work.

1.2. Notations and mathematical preliminaries

We denote the set of real numbers and positive integers, respectively, by $\mathbb{R}$ and $\mathbb{Z}_{\geq 1}$. For a vector $x \in \mathbb{R}^n$, where $n \in \mathbb{Z}_{\geq 1}$, with components $x_i \in \mathbb{R}$, $i \in \{1, \dots, n\}$, we use the norm $\|x\| = \sum_{i=1}^{n} |x_i|$. The set of all $n \times n$ matrices is denoted by $\mathbb{R}^{n\times n}$.

We recall some basic notions of graph theory from Bondy and Murty (2008). A *directed graph*, or simply *digraph*, is a pair $\mathcal{G} = (V, \mathcal{E})$, where V is a finite set called the vertex set and $\mathcal{E} \subseteq V \times V$ is the edge set. If V has n elements, we say that $\mathcal{G}$ is of order n which, unless otherwise noted, is the standard assumption throughout the paper. We can associate an adjacency matrix $\mathsf{A} \in \mathbb{R}^{n\times n}$ to $\mathcal{G}$, which has the property that the entry $a_{ij} \neq 0$, $i, j \in \{1, \dots, n\}$, if $(v_i, v_j) \in \mathcal{E}$ and $a_{ij} = 0$, otherwise. We say that $\mathcal{G} = (V, \mathcal{E})$ is *undirected*, or simply *graph* if $\mathcal{E}$ consists of unordered pairs of vertices. Given an edge $(u, v) \in \mathcal{E}$, we call u the *tail* and v is called the *head*. We say that u an *in-neighbor* of v and v an *out-neighbor* of u; for an undirected graph, we simply say that u and v are neighbors. Given an undirected graph $\mathcal{G} = (V, \mathcal{E})$, two edges $(u_1, v_1) \in \mathcal{E}$ and (u_2, v_2) are called adjacent if they have a vertex in common. A *matching* for an undirected graph is a set of pairwise nonadjacent edges. For a directed graph, a matching is matching of its underlying undirected graph, i.e., the graph with the same number of vertices obtained by turning any directed edge to an undirected one.

A digraph $\mathcal{G}_1 = (V_1, \mathcal{E}_1)$, where $E_1 \subseteq V_1 \times V_1$, is a subgraph (denoted by $\mathcal{G}_1 \subset \mathcal{G}$) of $\mathcal{G} = (V, \mathcal{E})$ if $V_1 \subset V$ and $\mathcal{E}_1 \subset \mathcal{E}$. Two subdigraphs $\mathcal{G}_1, \mathcal{G}_2 \subset \mathcal{G}$ are disjoint if they have no vertex in common. We say that $\mathcal{G}_1 \dots, \mathcal{G}_m$ is a *(vertex) decomposition* of $\mathcal{G}$ if they are pairwise disjoint and their union is $\mathcal{G}$. A *directed path* in a digraph, or in short path, is an ordered sequence of vertices so that any two consecutive vertices in the sequence are an edge of the digraph. A *cycle* in a digraph is a directed path

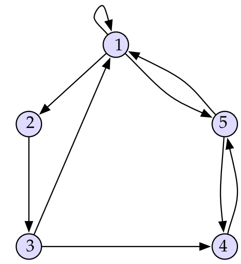

Fig. 1. The digraph associated to the sparse matrix system (1).

that starts and ends at the same vertex and has no other repeated vertex. The notion of a Hamiltonian subdigraph, which we recall from Belabbas (2013), plays an important role in this paper. A *Hamiltonian cycle* is a cycle that visits every vertex exactly once. A Hamiltonian (vertex) decomposition of $\mathcal{G}$ is a decomposition of $\mathcal{G}$ into disjoint subdigraphs, where each subdigraph admits a Hamiltonian cycle. A *Hamiltonian subdigraph* is a subdigraph of $\mathcal{G}$ that admits a Hamiltonian decomposition. Finally, a Hamiltonian k-subdigraph is a Hamiltonian subdigraph with $k \in \mathbb{Z}_{\geq 1}$ vertices.

2. Problem statement

Consider the vector space of matrices Σ_α in $\mathbb{R}^{n\times n}$, where $\alpha \subset \{1, \dots, n\} \times \{1, \dots, n\}$, and all entries not in α are forced to be zero. We refer to such vectors spaces as the *sparse matrix systems* (or sparse matrix spaces as in Belabbas, 2013), where the term *system* here refers to the linear system of differential equations that can be assigned to a given matrix in Σ_α, c.f. (2). We shall say that Σ_α is a *stable sparse matrix system* if Σ_α contains a Hurwitz matrix. We next provide an example to motivate the problem under study in this paper.

Let $\Sigma_\alpha \subset \mathbb{R}^{5\times 5}$ be the sparse matrix system given by

$$\Sigma_\alpha = \begin{pmatrix} \star & \star & 0 & 0 & \star \\ 0 & 0 & \star & 0 & 0 \\ \star & 0 & 0 & \star & 0 \\ 0 & 0 & 0 & 0 & \star \\ \star & 0 & 0 & \star & 0 \end{pmatrix}, \tag{1}$$

where the free parameters in α are denoted by $\star$s. As shown in Belabbas (2013, Section 2.2), Σ_α given by (1) is not a stable sparse matrix system, because its corresponding digraph shown in Fig. 1, where edges correspond to the free parameters, does not satisfy the necessary conditions for stability of sparse matrix systems (Belabbas, 2013, Theorem 2); let us recall this result, as we will be referring to this frequently throughout this note.

Theorem 1 (*Belabbas, 2013*)**.** *A sparse matrix system is stable only if its associated digraph contains a Hamiltonian k-subdigraph, for all $k \in \{1, \dots, n\}$. Moreover, if the associated digraph $\mathcal{G}$ contains a sequence of nested Hamiltonian subdigraphs $\mathcal{G}_1 \subset \mathcal{G}_2 \dots \subset \mathcal{G}_{n-1} \subset \mathcal{G}$ then it is stable.*

In particular, the digraph shown in Fig. 1 does not contain a Hamiltonian 4-subdigraph and hence Σ_α is not stable, i.e., there exists no choice of real numbers for the values of the free parameters that makes the system

$$\dot{x}(t) = A_\alpha x(t), \tag{2}$$

where $A_\alpha \in \Sigma_\alpha$ and $x(t) \in \mathbb{R}^n$, for $t \geq 0$, globally asymptotically stable to the origin (where $n = 5$ for this example). One, however, might still wonder if it is possible to use some control inputs that change the values of the free parameters on some of the edges over