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Brief paper

EM-based identification of continuous-time ARMA Models from irregularly sampled data ☆

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ABSTRACT

In this paper we present a novel algorithm for identifying continuous-time autoregressive moving-average models utilizing irregularly sampled data. The proposed algorithm is based on the expectation–maximization algorithm and obtains maximum-likelihood estimates. The proposed algorithm shows a fast convergence rate, good robustness to initial values, and desirable estimation accuracy. Comparisons are made with other algorithms in the literature via numerical examples.

1. Introduction

Identification of stochastic continuous-time (CT) models utilizing sampled data has been a recurring topic in different areas of research such as: System Identification, Time Series Analysis, and Econometrics (see *e.g.* Bergstrom, 1988, Elerian, 2008, Sinha & Rao, 1991, Tsai & Chan, 2005 and the references therein).

There are two approaches to identify stochastic continuous-time models (Larsson, Mossberg, & Söderström, 2008)

- **Indirect approach:** where an equivalent discrete-time (DT) model is identified and then transformed to the underlying continuous-time model. This approach has the advantage that well-established discrete-time identification methods can be utilized. It is well known that provided that the sampling interval is sufficiently small, then the mapping from a linear discrete-time model to the linear continuous-time model is one-to-one (Ding, Qiu, & Chen, 2009). On the other hand, the indirect approach cannot be directly used for irregular sampling since the equivalent discrete-time model is time-variant.
- **Direct approach:** where the continuous-time model is directly identified. Several direct identification methods utilize some approximations for the signal derivatives with respect to time. The estimates obtained by this approach are typically biased with a bias proportional to the sampling interval, see *e.g.* Larsson et al. (2008).

In many modern systems one can use much faster sampling rates than were previously possible (Goodwin, Agüero, Cea-Garrido, Salgado, & Yuz, 2013). In this case, one can develop identification algorithms utilizing the discrete-time model (similar to the indirect approach), but the continuous-time model is directly obtained by utilizing a re-parametrization of the system in terms of the *delta*-operator (see Goodwin et al., 2013 for details). An identification method for continuous-time models written in the state-space form utilizing data obtained by using fast sampling has been recently presented in Yuz, Alfaro, Agüero, and Goodwin (2011). The results in this paper have been extended in Aguilera, Godoy, Agüero, Goodwin, and Yuz (2014) in order to identify models in a class of hybrid systems where the state-space matrices are time-variant. On the other hand, in Chen,

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Garnier, Gilson, Agüero, and Godoy (2014) we have developed an algorithm to identify continuous-time autoregressive moving-average (CARMA) models utilizing data obtained by using fast sampling.

In this paper, we propose a method that extends the results in Aguilera et al. (2014), Chen et al. (2014) and Yuz et al. (2011) in order to consider CARMA models when slow sampling rates (compared to the system bandwidth) are utilized. Note that the aforementioned methods in Aguilera et al. (2014) and Yuz et al. (2011) cannot be directly applied to a CARMA model since it has a noise-free output equation when written in the state-space form.

1.1. Notation

$\boldsymbol{x}_i(t_k)$ is the ith component of the state vector $\boldsymbol{x}(t_k)$. $\boldsymbol{\theta}_i$ is the ith component of the vector of parameters $\boldsymbol{\theta}$. $\hat{\boldsymbol{\theta}}^r$ represents the rth iteration corresponding to an iterative estimation procedure (*e.g.* Newton-based, or expectation–maximization-based optimization algorithms) to estimate a vector of parameters $\boldsymbol{\theta}$. F_k represents the exponential matrix obtained when irregular sampling is utilized, *i.e.* $F_k = \exp\{A(t_{k+1} - t_k)\}$. $\dot{F}_{k,i}$ represents the derivative of the matrix F_k with respect to the ith component of the vector $\boldsymbol{\theta}$, *i.e.* $\dot{F}_{k,i} = \frac{\partial F_k}{\partial \theta_i}$. $\dot{Q}_{k,i}$ and $\dot{R}_{k,i}$ are similarly defined. The set of data collected from time t_1 to time t_k is denoted by $\mathcal{Y}_k = \{y(t_1), \dots, y(t_k)\}$. The set of states $\mathcal{X}_k$ is similarly defined, *i.e.* $\mathcal{X}_k = \{\boldsymbol{x}(t_1), \dots, \boldsymbol{x}(t_k)\}$. $|A|$ denotes the determinant of the matrix A.

1.2. Statement of the problem

In this paper we focus on the identification of CARMA models given by

$$\mathcal{D}(s)y(t) = \mathcal{C}(s)e(t) \tag{1}$$

where $e(t)$ is zero mean continuous-time white noise (CTWN) with spectral power $\mathrm{E}\{e^2(t)\} = \sigma^2$ assumed to be Gaussian. Note that CTWN has infinite variance and therefore remains a mathematical conjecture in the literature, a formal description is provided via the Wiener process, the interested reader is referred to *e.g.* Øksendal (2003) for more details on this topic. $\mathcal{D}(s)$ and $\mathcal{C}(s)$ are polynomials in the Laplace operator s

$$\mathcal{D}(s) = s^n + d_1 s^{n-1} + \dots + d_n \tag{2}$$

$$\mathcal{C}(s) = c_0 s^m + c_1 s^{m-1} \dots + c_m \tag{3}$$

where n and m are the degrees of $\mathcal{D}(s)$ and $\mathcal{C}(s)$, respectively. To ensure that $y(t)$ has finite variance we let $n > m$. Since $\{c_0, \dots, c_m\}$ can represent the gain of the system, the spectral power of $e(t)$ is fixed as a constant ($\sigma^2 = 1$). It is assumed that samples of $y(t)$ are collected irregularly at $\{t_1, t_2, \dots, t_N\}$, and $h_k = t_{k+1} - t_k$ denotes the irregular sampling interval. The identification objective is to estimate the unknown model parameters $\{d_1, \dots, d_n, c_0, \dots, c_m\}$ from the irregularly sampled data $\mathcal{Y}_N = \{y(t_1), \dots, y(t_N)\}$.

The layout of the remainder of the paper is as follows. In Section 2 we present the system of interest in the state-space form. In Section 3 we discuss relevant issues regarding maximum-likelihood estimation of CARMA model parameters utilizing irregular sampling. An identification method based on the expectation–maximization algorithm is developed in Section 4. Numerical examples are shown in Section 5 to illustrate the benefits of the proposed method. Finally, in Section 6, we draw conclusions.

2. State-space equivalent model

It is well known that the model in (1) can also be represented in the state-space form Gevers (2006)

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + Ke(t) \tag{4}$$

$$y(t) = C\boldsymbol{x}(t). \tag{5}$$

In addition, the sampled-data model corresponding to (4)–(5) with samples taken at $\{t_1, t_2, \dots, t_N\}$ is given by Söderström (2002)

$$\boldsymbol{x}(t_{k+1}) = F_k\boldsymbol{x}(t_k) + \boldsymbol{w}(t_k) \tag{6}$$

$$y(t_k) = C\boldsymbol{x}(t_k) \tag{7}$$

where $F_k = \exp\{A(t_{k+1} - t_k)\}$, $\boldsymbol{w}(t_k)$ is zero mean DT white noise with covariance matrix given by

$$Q_k = \mathbb{E}\{\boldsymbol{w}(t_k)\boldsymbol{w}^T(t_k)\} = \int_0^{h_k} \exp\{A\tau\}KK^T \exp\{A^T\tau\}d\tau \tag{8}$$

where $h_k = t_{k+1} - t_k$. Then, the DT stochastic process $\{y(t_k)\}$ has the same second order properties as $\{y(t)\}$ at the sampling instants $\{t_1, \dots, t_N\}$. The initial state $x(t_1)$ is assumed to have a normal distribution $\boldsymbol{x}(t_1) \sim \mathcal{N}(\mu_1, P_1)$.

Note that KK^T is singular. However, the covariance matrix Q_k is generally non-singular (Söderström, 2002, p. 88). In fact, the matrix Q_k is non-singular for any value of h_k if the pair $[-A, K]$ is controllable (Wolowich, 1974, p. 66).

Several identification algorithms for continuous-time autoregressive (CAR), continuous-time autoregressive exogenous (CARX), and CARMA models utilize the state-space model in (4)–(5). For example in Larsson, Mossberg, and Söderström (2007) an observable canonical form is utilized to obtain the Cramer–Rao lower bound corresponding to the estimates of a CARX process where the exogenous signal is considered as the output of a CARMA model.

In this paper we propose to use a general parametrization for the state-space matrices $A = A(\boldsymbol{\theta}), K = K(\boldsymbol{\theta}), C = [1\,0\,\cdots\,0]$. This parametrization not only covers the matrices corresponding to traditional canonical forms but also covers the fully parametrized matrices A and K. As suggested in McKelvey and Helmersson (1996), one should use an over-parametrized state-space model in order to overcome the numerical issues that arise when using canonical parametrizations in an optimization-based estimation algorithm. In addition, over-parametrized state-space matrices have been utilized in Agüero, Tang, Yuz, Delgado, and Goodwin (2012), Gibson and Ninness (2005) and Yuz et al. (2011). Note that in the parametrization utilized in this paper, the matrix C does not contain parameters to be estimated. This choice will become clear once the proposed method is presented in Section 4.

3. Maximum-likelihood estimation

In maximum-likelihood (ML) estimation, the log-likelihood function is defined to be the logarithm of the probability density function of output observations parametrized with a vector of parameters $\boldsymbol{\theta}$, *i.e.*

$$l(\boldsymbol{\theta}) = \log p\left(\mathcal{Y}_N|\boldsymbol{\theta}\right). \tag{9}$$

The ML estimate of $\boldsymbol{\theta}$ is defined as the solution of the following optimization problem

$$\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} l(\boldsymbol{\theta}). \tag{10}$$

It is well known that the value of the log-likelihood function (hereafter shortened as likelihood function) can be computed by prediction error decomposition via the Kalman filter (Agüero et al., 2012; Ljung, 1999; Söderström & Stoica, 1989)

$$l(\boldsymbol{\theta}) = -\frac{1}{2}\sum_{k=1}^{N} \epsilon_k^T \Lambda_k^{-1}\epsilon_k - \frac{1}{2}\sum_{k=1}^{N}\log|\Lambda_k| \tag{11}$$