Automatica 77 (2017) 14–17

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Technical communique

Further refinement on controller design for linear systems with input saturation☆

Do Wan Kim [1]

Hanbat National University, Republic of Korea

ARTICLE INFO

Article history:
Received 8 February 2016
Received in revised form
13 August 2016
Accepted 16 October 2016

Keywords:
Saturation
Linear matrix inequality (LMI)
Lyapunov
Stabilization
Exponential stability
Local stability

ABSTRACT

This paper addresses on the design problem for a class of continuous-time linear systems under the saturated linear feedbacks. The concerned problem becomes an exponential stabilization one through the further refined treatment on the saturation nonlinearity, including a property of absolute value and the $\mathcal{S}$-procedure lemma. Then, sufficient linear matrix inequality (LMI) conditions for its local exponential stabilizability are derived in the sense of Lyapunov stability criterion. A numerical example is provided to show the reduced conservatism of the proposed design conditions.

1. Introduction

Control input saturation is unavoidable in implementing the practical control systems because of the magnitude limitation of the input. Unless the nonlinearity of the saturation is reflected in design, it may cause the degraded control performance or even the instability (Kapila & Grigoriadis, 2002). Therefore, the analysis and design of the saturated control system has been widely investigated (see Alamo, Cepeda, & Limon, 2005; Cao & Lin, 2003; Hindi & Boyd, 1998; Hu & Lin, 2001; Hu, Lin, & Chen, 2002; Kapila & Grigoriadis, 2002; Khalil, 2002; Li & Lin, 2013; Lin & Saberi, 1995; Tarbouriech & Gouaisbaut, 2012; Tarbouriech, Prieur, & Da Silva, 2006; Wei, Zheng, & Xu, 2015a,b; Zhou, Zheng, & Duan, 2011 and references therein).

Most of the prior works (Alamo et al., 2005; Cao & Lin, 2003; Hindi & Boyd, 1998; Hu & Lin, 2001; Hu et al., 2002; Khalil, 2002; Li & Lin, 2013; Tarbouriech & Gouaisbaut, 2012; Tarbouriech et al., 2006; Zhou et al., 2011) have pursed the local stability or stabilization, since it is usually required to some stability assumption on the open-loop system for the global results (Lin & Saberi, 1995), and hence the estimation of the domain of attraction of the closed-loop system is one of important issues. A distinct difference among their conservatism arises from how they handle the saturation nonlinearity (Zhou et al., 2011). Its treatment can be classified into two main approaches: the first one is to treat the troublesome saturation as a locally sector bounded nonlinearity (see, Hindi & Boyd, 1998; Khalil, 2002; Tarbouriech & Gouaisbaut, 2012; Tarbouriech et al., 2006 and references therein), while the other is to represent it as a (polytopic or linear) differential inclusion (see, Alamo et al., 2005; Cao & Lin, 2003; Hu & Lin, 2001; Hu et al., 2002; Li & Lin, 2013; Zhou et al., 2011 and references therein).

Especially the differential inclusion approach (Hu et al., 2002) provided a necessary and sufficient condition for an ellipsoid to be invariant in the single input. Namely, the further relaxation is unnecessary in this case. However, the relaxed conservatism in the multi inputs is still an important and open topic of this field although some significant improvements can be made by introducing more free variables (Alamo et al., 2005; Zhou et al., 2011) and a composite Lyapunov function (Li & Lin, 2013).

This study aims to provide design conditions for a class of continuous-time linear systems subject to input saturation. The concerned design problem is tackled by a regional exponential stabilization under the saturated linear state feedback. The proposed treatment on the saturation nonlinearity comes from the

☆ This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT & Future Planning (NRF-2014R1A1A1004610). The material in this paper was not presented at any conference. This paper was recommended for publication in revised form by Associate Editor Tingshu Hu under the direction of Editor André L. Tits.

E-mail address: dowankim@hanbat.ac.kr.

[1] Fax: +82 42 821 1088.

http://dx.doi.org/10.1016/j.automatica.2016.11.004

property of saturation function (see, Alamo et al., 2005, Property 4), which enables an analyzable upper bound of the saturation function. However, through introducing the inequality property of absolute value and $\mathcal{S}$-procedure Lemma (Yakubovich, 1971) (rather than the technique used in Alamo et al., 2005, Properties 1 and 2, namely, additional conditions related to large amount of decision variables), the treatment is further refined in a less conservative manner. Based on this refined treatment of the saturation nonlinearity, this paper provides sufficient linear matrix inequality (LMI) conditions for a regional exponential stabilization in the sense of Lyapunov stability criterion. Finally, examples are given to demonstrate the reduced conservativeness of the proposed design methodology.

Notations: The relation $P \succ Q$ ($P \prec Q$) means that the matrix $P-Q$ is positive (negative) definite. $\lambda_{max}(A)$ ($\lambda_{min}(A)$) is the maximum (minimum) eigenvalue of a symmetric matrix A. $A_{(i)}$ and $A^{(i)}$ denote the ith row and column of the matrix A, respectively. $\mathcal{I}[1, m]$ indicates the integer set $\{1, \ldots, m\}$ with $m \in \mathbb{R}_{>1}$. An ellipsis is adopted for long symmetric matrix expressions, e.g.,

$$K^T \begin{bmatrix} \text{sym}\{S\} & (*) \\ M & Q^T(*) \end{bmatrix} (*) := K^T \begin{bmatrix} S+S^T & M^T \\ M & Q^TQ \end{bmatrix} K.$$

2. Problem formulation

Consider the linear system with the input saturation

$$\dot{x} = Ax + B\text{sat}(u) \tag{1}$$

where $x \in \mathbb{R}^n$ the state; $u \in \mathbb{R}^m$ the input; sat : $\mathbb{R}^m \to \mathbb{R}^m$ the standard saturation function with the unity saturation level defined as $\text{sat}(u) := [\text{sat}(u_1)\,\text{sat}(u_2)\cdots\text{sat}(u_m)]^T$ in which $\text{sat}(u_i) := \text{sign}(u_i)\min\{|u_i|, 1\}$. Here, $\text{sat}(\cdot)$ is abused to denote both the scalar and the vector valued functions.

Remark 1. Note that nonunity saturation level can be absorbed into B and u. For example, consider

$$\dot{x} = Ax + \tilde{B}\text{sat}_{\bar{u}}(\tilde{u}) \tag{2}$$

where $\text{sat}_{\bar{u}_i}(\tilde{u}_i) := \text{sign}(\tilde{u}_i)\min\{|\tilde{u}_i|, \bar{u}_i\}$ with $\bar{u}_i \in \mathbb{R}_{>0}$. Let $B := \tilde{B}\bar{U}$ and $u := \bar{U}^{-1}\tilde{u}$ with $\bar{U} := \text{diag}\{\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_m\}$. Then, (2) becomes (1) (for more details, see Cao & Lin, 2003).

Our concerned problem is formulated as follows.

Problem 1. Consider (1). Our problem is to determine a state feedback controller

$$u = Kx \tag{3}$$

and a contractively invariant set $\Omega_c := \{x \in \mathbb{R}^n : x^TPx < c\}$ with $P = P^T \succ 0$ and $c \in \mathbb{R}_{>0}$ such that for $x(t_0) \in \Omega_c$, the closed-loop system of (1) and (3) is exponentially stable at the origin with a contractively invariant set Ω_c contained in the domain of attraction.

The following lemmas are the widely-used sufficient conditions for Problem 1:

Lemma 1. *For $x(t_0) \in \Omega_1$, the closed-loop system of (2) and (3) is asymptotically stable at the origin with a contractively invariant set Ω_1 contained in the domain of attraction if there exist $\tilde{P} = \tilde{P}^T \succ 0$, and $\tilde{K}$ such that*

$$\text{sym}\left\{A\tilde{P} + B\tilde{K}\right\} \prec 0$$

$$\begin{bmatrix} -\bar{u}_i^2 & * \\ \tilde{K}_{(i)}^T & -\tilde{P} \end{bmatrix} \prec 0, \quad \forall i \in \mathcal{I}[1, m]$$

In this case, $P = \tilde{P}^{-1}$ and $K = \tilde{K}\tilde{P}^{-1}$.

Proof. The proof directly follows from the exponential stabilizability and $\Omega_1 \subset \{u_i \in \mathbb{R} : |u_i| < \bar{u}_i\}$ (for example, see Du & Zhang, 2009).

Lemma 2 (*Alamo et al., 2005*). *For $x(t_0) \in \Omega_1$, the closed-loop system of (1) and (3) is asymptotically stable at the origin with a contractively invariant set Ω_1 contained in the domain of attraction if there exist $\tilde{P} = \tilde{P}^T \succ 0$, $\tilde{K}$, and Y^U such that*

$$\text{sym}\left\{A\tilde{P} + \sum_{i\in U^c} B^{(i)}\tilde{K}_{(i)} + \sum_{i\in U} B^{(i)}Y_{(i)}^U\right\} \prec 0$$

$$\begin{bmatrix} -\tilde{P} & * \\ Y_{(i)}^U & -1 \end{bmatrix} \prec 0, \quad \forall i \in U$$

where U are all subsets of the set of integers $\mathcal{I}[1, m]$, for example, if $m = 2$, then $U \subseteq \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$, and U^c denotes the complementary of U in $\mathcal{I}[1, m]$. In this case, $P = \tilde{P}^{-1}$ and $K = \tilde{K}\tilde{P}^{-1}$.

Proof. The proof directly follows from Alamo et al. (2005, Theorem 1).

Remark 2. Lemma 1 is the well-known result on adding the input constraint $\Omega_1 \subset \{u_i \in \mathbb{R} : |u_i| < \bar{u}_i\}$ for all $i \in \mathcal{I}[1, m]$ to avoid the saturation function $\text{sat}(u)$. However, this constraint brings conservatism into the design. That is, it is difficult to obtain the possibly large Ω_c. Besides, the control gain in design is subject to the magnitude restriction due to $|u_i| < \bar{u}_i$. Lemma 2 provides a less conservative result in the multi-input case than prior works (Hu & Lin, 2001; Hu et al., 2002) based on linear differential inclusions by introducing slack variables Y^U. However, as discussed in Alamo et al. (2005), Lemma 2 can be limited because of the exponential number of decision variables.

3. Main results

Before proceeding to our main results, the following definitions and lemmas will be needed throughout the proof:

Definition 1 (*Hu & Lin, 2001*). Define the set $\mathcal{D}$ composed of 2^m different diagonal matrices $D_i \in \mathbb{R}^{m\times m}$, $i \in \mathcal{I}[1, 2^m]$ in which their diagonal elements are taken as the value of either 1 or 0. Furthermore, $D_i^- := I - D_i$.

Definition 2. Consider Definition 1. Define the set $\bar{\mathcal{D}}_i := \{D_j \in \mathcal{D} | D_iD_j = 0, \forall j \in \mathcal{I}[1, 2^m]\} := \{\bar{D}_{i1}, \ldots, \bar{D}_{i2^{\bar{m}}}\}$, $i \in \mathcal{I}[1, 2^m]$, where $\bar{m}(i) = \text{Tr}(D_i^-)$. Furthermore, $\bar{D}_{ij}^- := I - D_{ij}$.

Remark 3. For example, if $m = 2$, then $\mathcal{D} := \{D_1, D_2, D_3, D_4\}$ with $D_1 = \text{diag}\{0, 0\}$, $D_2 = \text{diag}\{1, 0\}$, $D_3 = \text{diag}\{0, 1\}$, $D_4 = \text{diag}\{1, 1\}$. In this case, $\bar{\mathcal{D}}_1 = \{D_1, D_2, D_3, D_4\} := \{\bar{D}_{11}, \bar{D}_{12}, \bar{D}_{13}, \bar{D}_{14}\}$, $\bar{\mathcal{D}}_2 = \{D_1, D_3\} := \{\bar{D}_{21}, \bar{D}_{22}\}$, $\bar{\mathcal{D}}_3 = \{D_1, D_2\} := \{\bar{D}_{31}, \bar{D}_{32}\}$, and $\bar{\mathcal{D}}_4 = \{D_1\} := \{\bar{D}_{41}\}$.

Lemma 3 (*Alamo et al., 2005*). *Given $a, b \in \mathbb{R}$, then*

$$a\text{sat}(b) \leqslant \max\{ab, -|a|\}.$$

Lemma 4. *Given $a \in \mathbb{R}$, then*

$$-|a| \leqslant \max\{-a^2, -1\}.$$

Proof. It is true that if $|a| \leqslant 1$, then $-1 \leqslant -|a| \Leftrightarrow -|a| \leqslant -a^2$; otherwise, $-|a| < -1$.

Lemma 5. *Given $a \in \mathbb{R}$, then*

$$a\text{sat}(b) \leqslant \max\{ab, -a^2, -1\}.$$