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Analysis of a new stabilized higher order finite element method for advection–diffusion equations

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Abstract

We consider a singularly perturbed advection–diffusion two-point boundary value problem whose solution has a single boundary layer. Based on piecewise polynomial approximations of degree $k \geqslant 1$, a new stabilized finite element method is derived in the framework of a variation multiscale approach. The method coincides with the SUPG method for $k = 1$ but differs from it for $k \geqslant 2$. Estimates for the error to an appropriate interpolant are given in several norms on different types of meshes. For $k = 1$ enhanced accuracy is achieved by superconvergence. Postprocessing guarantees the same estimates for the error to the solution itself.

Keywords: Stabilized finite element method; Singular perturbation; Advection–diffusion equation; Shishkin mesh; Superconvergence; Postprocessing

1. Introduction

We consider the two-point boundary value problem

$$-\varepsilon u'' + b(x)u' + c(x)u = f(x) \quad \text{in } (0,1), \quad u(0) = u(1) = 0, \tag{1}$$

with sufficiently smooth functions b, c, f, and a small positive parameter $0 < \varepsilon \ll 1$. We assume that

$$c(x) - \frac{1}{2}b'(x) \geqslant \gamma > 0, \quad x \in [0,1], \tag{2}$$

which guarantees the unique solvability of the problem.

Standard Galerkin-type finite element methods exhibit spurious oscillations unless the mesh is very fine. Therefore, a number of stabilized methods (SUPG, Galerkin-least squares, residual free bubble, etc.) have been developed and extended both to the multi-dimensional case and to the incompressible Navier–Stokes equations; see [4] for the beginning and [15,16] for a survey. The variational multiscale method [8,9] has been introduced as a framework for a better understanding of the fine-to-coarse scale effects and as a platform for the development of new numerical methods. Recently, Hughes and Sangalli [10,11] succeeded in giving explicit formulas for the fine-scale Green's function arising in variational multiscale analysis. An important observation of their approach is that the fine-scale problem for higher order finite element approximations can be considered as a constrained bubble problem.

For the constant coefficient case (with $c = 0$) and piecewise linear finite elements the close relations between the SUPG method, the residual free bubble approach, and the variational multiscale method are well-known. However, when using higher order finite elements the variational multiscale approach leads to a new stabilized method which seems to be not analyzed up to now. In order to distinguish this new method from the SUPG method we will call it variational multiscale (VMS) method.

The main objective of the paper is to analyze the VMS method and to give error estimates in several norms on different types of meshes. In Section 2 we shortly describe the variational multiscale approach leading to the new numer-

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ical method and show an alternative way for its derivation. Then, in Section 3 we give an error estimate in a mesh dependent norm which is related to the discrete bilinear form. It is important to note that in the case of higher order finite elements a new interpolation has been used to get the desired error estimates. Section 4 is devoted to ε-uniform error estimates on families of Shishkin meshes. These estimates are based on a decomposition of the solution into a smooth and layer part, respectively, as well as a detailed study of approximation properties in and outside the layer region. It turns out that the error of the new VMS method to the interpolant is of order $k+1/2$ uniformly in ε for piecewise polynomials of degree k. Using superconvergence properties in the case $k=1$ the accuracy can be enhanced to almost second order. Finally, we show that by a proper postprocessing the same bounds can be established for the error of the postprocessed numerical solution to the solution itself.

Notations. Throughout the paper C will denote a generic positive constant that is independent of ε and the mesh.

We use the standard Sobolev spaces $W^{k,p}(D)$, $H^k(D)=W^{k,2}(D)$, $H^k_0(D)$, $L^p(D)=W^{0,p}(D)$ for nonnegative integers k and $1\leqslant p\leqslant\infty$ and write $(\cdot,\cdot)_D$ for the $L^2(D)$ inner product. Here D is any measurable subset of $(0,1)$. Then, $|\cdot|_{k,p,D}$ and $\|\cdot\|_{k,p,D}$ are the usual Sobolev seminorm and norm on $W^{k,p}(D)$. When $D=(0,1)$ we drop D from the notation for simplicity. We will also simplify the notation in the case $p=2$ by setting $\|\cdot\|_{k,D}=\|\cdot\|_{k,2,D}$ and $|\cdot|_{k,D}=|\cdot|_{k,2,D}$.

2. Derivation of the method

2.1. Variational multiscale method

The weak formulation of (1) is given by

Find $u\in V:=H^1_0(0,1)$ such that for all $v\in V$

$$a(u,v):=\varepsilon(u',v')+(bu'+cu,v)=(f,v). \tag{3}$$

The idea of the variational multiscale approach is to split the solution space V into resolvable and unresolvable scales. This is realized by choosing a finite element space V_h which represents the resolvable scales and a projection operator $P:V\to V_h$ such that

$$V=V_h\oplus V^\diamondsuit,\quad u=Pu+(I-P)u=u_h+u^\diamondsuit.$$

Now the weak formulation (3) can be reformulated as

Find $u_h\in V_h$ and $u^\diamondsuit\in V^\diamondsuit$ such that

$$a(u_h+u^\diamondsuit,v_h)=(f,v_h)\quad\forall v_h\in V_h, \tag{4}$$

$$a(u_h+u^\diamondsuit,v^\diamondsuit)=(f,v^\diamondsuit)\quad\forall v^\diamondsuit\in V^\diamondsuit. \tag{5}$$

In order to remove the unresolvable scales we define $M(u_h)$ and $F(f)$ as the solutions of the problems

Find $M(u_h)\in V^\diamondsuit$, $F(f)\in V^\diamondsuit$ such that

$$\begin{aligned}&a(M(u_h),v^\diamondsuit)=-a(u_h,v^\diamondsuit),\\&a(F(f),v^\diamondsuit)=(f,v^\diamondsuit)\quad\forall v^\diamondsuit\in V^\diamondsuit.\end{aligned} \tag{6}$$

Then, the solution of (5) becomes $u^\diamondsuit=M(u_h)+F(f)$ and the elimination in (4) leads to the *stabilized method*:

Find $u_h\in V_h$ such that for all $v_h\in V_h$

$$a(u_h+M(u_h),v_h)=(f,v_h)-a(F(f),v_h). \tag{7}$$

Remark 1. Note that the variational multiscale approach is not restricted to the one-dimensional case but can also be used for an arbitrary variational problem defined by a coercive, continuous bilinear form on a Hilbert space V.

Remark 2. The problem (7) is finite dimensional, however it is based on the solution of the infinite dimensional problems (6). Therefore in practice one has to approximate the mappings M and F [1–3]. In special situations (e.g., one-dimensional case, piecewise constant coefficients) one can obtain explicit representations for these mappings.

Let $0=x_0<x_1<\cdots<x_N=1$ be a partition $\mathcal{T}_h$ of $[0,1]$. We denote an arbitrary subinterval (x_{i-1},x_i) by K, its length by $h_K=x_i-x_{i-1}$, and set $h=\max_{K\in\mathcal{T}_h}h_K$. We consider two examples of the variational multiscale approach. First let V_h be the space of piecewise linear finite elements and $P:V\to V_h$ the $H^1_0(0,1)$-projection given by

$$((Pv)',w_h')=(v',w_h')\quad\forall w\in V_h.$$

It can be shown (see Section 3) that $(Pv)(x_i)=v(x_i)$, $i=0,\ldots,N$, that means, P is the piecewise linear nodal interpolation. As a consequence, $V^\diamondsuit$ becomes the bubble space

$$V^\diamondsuit=\bigoplus_{K\in\mathcal{T}_h}H^1_0(K)$$

and the problems (6) can be solved locally on each element K. If, in addition, $c=0$ and the functions b,f are piecewise constant, we can find explicit representations of the operators M and F. This leads to the stabilized method (7) which is equal to the SUPG method

Find $u_h\in V_h$ such that for all $v_h\in V_h$

$$a(u_h,v_h)+\sum_{K\in\mathcal{T}_h}\tau_K(bu_h',bv_h')_K=(f,v_h)+\sum_{K\in\mathcal{T}_h}\tau_K(f,bv_h')_K$$

with the SUPG parameter

$$\tau_K=\frac{h_K}{2b}\left(\coth q_K-\frac{1}{q_K}\right),\quad q_K=\frac{bh_K}{2\varepsilon}.$$

In the second example let V_h be the space of piecewise quadratic finite elements and $P:V\to V_h$ be the $H^1_0(0,1)$-projection. In Section 3 we will see that now

$$(Pv)(x_i)=v(x_i),\quad i=0,\ldots,N,\quad\text{and}$$

$$\int_{x_{i-1}}^{x_i}(Pv-v)(x)\,\mathrm{d}x=0,\quad i=1,\ldots,N.$$

Because the quadratic bubble function $x\mapsto(x_i-x)(x-x_{i-1})$ belongs to $H^1_0(x_{i-1},x_i)$ the space of unresolvable scales is no longer the bubble space, instead it is the constrained bubble space

$$V^\diamondsuit=\left\{v^\diamondsuit\in\bigoplus_{K\in\mathcal{T}_h}H^1_0(K):Pv^\diamondsuit=0\right\}.$$