Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 196 (2007) 4027–4044
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

On deformational and configurational mechanics of micromorphic hyperelasticity – Theory and computation

C. Britta Hirschberger, Ellen Kuhl, Paul Steinmann *

Department of Mechanical Engineering, Chair of Applied Mechanics, University of Kaiserslautern, P.O. Box 3049, D-67653 Kaiserslautern, Germany

Received 6 June 2006; received in revised form 20 November 2006; accepted 26 February 2007
Available online 8 May 2007

Abstract

A micromorphic continuum formulation is presented in the context of both, the spatial- and the material-motion problem. For both approaches the kinematics as well as the balance relations together with the various representations of the occurring stress fields are derived. The relations between the spatial-motion problem and the material-motion problem quantities are examined in detail. Upon a hyperelastic constitutive assumption a finite-element approximation is derived and the material–force method, which is especially suited for defect-mechanics problems, is successfully applied to the present micromorphic continuum theory.

Keywords: Micromorphic continua; Configurational mechanics; Material forces; Finite element method

1. Introduction

The micromorphic continuum theory is used to describe materials which possess a significant microstructure and therefore exhibit scale-dependent behaviour. These microstructures are viewed as so-called microcontina, which are assumed to be attached to each physical point and may experience both stretch and rotation deformations which are affine throughout the microcontinuum, nevertheless kinematically independent from the deformation on the macroscale.

The micromorphic continuum as a microcontinuum theory has first been introduced by Eringen [6,7] and is part of the group of so-called generalised continua for which the couple-stress theory of the brothers Cosserat and Cosserat [2] laid the foundation in the early 20th century. For related early developments the reader is referred to [31,32,22,13], just to mention a few. The so-called micropolar and microstretch continua may be considered as special cases of the micromorphic theory, since here specific constraints apply on the deformation of the microcontinuum. For instance in the case of the micropolar continuum, the microcontinuum may only experience rotation, see the contributions of [26,29,4] as well as those from the group of Tsakmakis (e.g., [9,3]), just to mention a few. A comprehensive overview on microcontinuum theories can be found in the contemporary monograph of Eringen [8]. Not only the different microcontinuum theories are congeneric, additionally, close relations between the latter and other nonlocal theories exist. Particularly the micromorphic and the second-order gradient theory can be transferred into each other by limit considerations, as has recently been shown by Kirchner and Steinmann [11]. This close relation allows to transfer the constitutive and finite-element formulations that were originally developed for micromorphic continua, to second-order gradient theories. As examples for such formulations of the gradient elasticity serve for instance the following contributions: Shu et al. [25] and Amanatidou and Aravas [1] for small deformations, as well as, extended to large deformations, Shu and Barlow [24] and – in the context of a homogenisation procedure [14–16].

* Corresponding author. Tel.: +49 631 205 2419; fax: +49 631 205 2128.
E-mail addresses: bhirsch@rhrk.uni-kl.de (C. Britta Hirschberger), ekuhl@rhrk.uni-kl.de (E. Kuhl), ps@rhrk.uni-kl.de (P. Steinmann).

0045-7825/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.cma.2007.02.015

Generally the deformation of continuum bodies can be described in two manners: One perspective is the so-called deformational mechanics and the other is the concept of configurational mechanics. Within deformational mechanics – or rather the *spatial-motion problem* – the spatial-motion of a physical point with particular material position is observed with the change of time – "*quo vadis*?". This description is characterised on the one hand by a parametrisation of the occurring quantities by the material position and on the other hand by the fact that these quantities belong to the spatial manifold (or are two-point tensors mapping towards the spatial manifold). Opposed to that, in the configurational-mechanics perspective, which we refer to as the *material-motion problem*, the spatial placement is fixed while the material origin of physical points passing by is the unknown quantity of observation – "*unde venis*?". The material-motion problem is formulated in material quantities (or two-point quantities mapping to the material manifold) which are parameterised by the spatial position.

In the spatial-motion problem we deal with the intuitively-known and widely-used spatial forces, which are defined as being energetically conjugate to variations in the spatial placement of physical points at fixed material position. Contrary to that, the perspective of the material-motion problem yields the so-called *material forces*. These are energetically conjugate to variations in the material position of a particular physical point. They are characterised in the literature to act as driving forces for the propagation of defects such as cracks and voids. Thus the material forces can for instance serve as criteria for crack propagation. For references on the concept of deformational vs. configurational mechanics in general and the material force method in particular, see for instance the contributions of Maugin and coworkers [5,19,20,23,21], Gurtin [10] or the group of Steinmann [27,28,30,17,18] as well as the references cite therein. It suggests itself to apply this dual perspective and within this context especially the material force method to the micromorphic continuum. From this, besides a deeper understanding of the theory, we strive for the predictive character of the material forces for fracture-mechanics considerations within the micromorphic continuum. Due to the relations shown in Kirchner and Steinmann [11], the full configurational-mechanics perspective on the second-order gradient theory given by [12], is especially helpful to establish the analogous perspective on the present micromorphic continuum.

Consequently, in this contribution a micromorphic continuum formulation is presented from the dual perspectives of both deformational and configurational mechanics. The kinematics is presented for finite deformations including the macro and the microcontributions. A restriction to isotropic hyperelasticity ensures the existence of a potential energy. For conservative static problems Dirichlet's principle results in the Euler–Lagrange equations, i.e., the balance of momentum in its strong form. Thus upon the definition of proper micromorphic kinematics, a variation of the potential energy renders the weak form of the balance of momentum together with Neumann boundary conditions, while the variations of primary kinematic variables have to satisfy the homogeneous Dirichlet boundary conditions. Based on the material-motion description, the material force method is applied, which, due to the existence of the microscale, bears the micromorphic material forces as well as additional higher-order quantities.

The article is structured as follows. In Section 2 the micromorphic continuum is introduced and its description is derived from a complete configurational-mechanics perspective. In Section 3 we establish a straightforward constitutive formulation which enables us to perform a finite-element approximation for the micromorphic continuum as presented in Section 4. Numerical examples incorporating parametric studies and the evaluation of the occurring material force quantities are computed in Section 5. The contribution closes with a short summary and a brief outlook on further research on the field.

2. The micromorphic continuum

As indicated before, the micromorphic continuum is described as a macrocontinuum of which each physical point is endowed with a microstructure referred to as the microcontinuum. These microcontinua may experience arbitrary deformations consisting of both stretch and rotation which are required to be affine, nevertheless kinematically independent from the macrocontinuum. A point $\mathscr{P}$ on the macroscale is described by the placement vectors $\boldsymbol{X}$ in the material configuration $\mathscr{B}_0$ and $\boldsymbol{x}$ in the spatial configuration $\mathscr{B}_t$. On the microscale, a point $\bar{\mathscr{P}}$ of the microcontinuum is denoted with the microplacement vectors $\bar{\boldsymbol{X}}$ in the material configuration $\bar{\mathscr{B}}_0$ and $\bar{\boldsymbol{x}}$ in the spatial configuration $\bar{\mathscr{B}}_t$, respectively.

Since a material point is equipped with a microstructure that is kinematically independent, additional balance equations besides the ordinary balance of momentum have to be considered for the micromorphic theory. These may be obtained by an energy consideration, which is here derived from Dirichlet's principle. Here the set of state variables is given by $\mathscr{S} = \{\boldsymbol{\varphi}, \bar{\boldsymbol{F}}\}$, wherein the first argument describes the macrostructural kinematics while the second characterises the deformation of the microcontinuum. A constitutive functional is to be found, which in the most general hyperelastic case incorporates the following dependencies: $\mathscr{C} = \mathscr{C}(\boldsymbol{\varphi}, \nabla_X \boldsymbol{\varphi}, \bar{\boldsymbol{F}}, \nabla_X \bar{\boldsymbol{F}};\ \boldsymbol{X})$.

First, the spatial-motion problem is presented in Section 2.1, before the material-motion problem described in Section 2.2. Both approaches are compared in Section 2.3. The following derivations of the spatial-motion problem follow the line of the finite-deformation part of Kirchner and Steinmann [11], while for the material-motion problem, parallels to the material settings of the second-order gradient theory as presented by Kirchner and Steinmann [12] can be recognised.