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Topological sensitivity analysis for three-dimensional linear elasticity problem

A.A. Novotny [a,*], R.A. Feijóo [a], E. Taroco [a], C. Padra [b]

[a] *Laboratório Nacional de Computação Científica LNCC/MCT, Av. Getúlio Vargas 333, 25651-075 Petrópolis – RJ, Brazil*
[b] *Centro Atómico Bariloche, 8400 Bariloche, Argentina*

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Abstract

In this work we use the Topological-Shape Sensitivity Method to obtain the topological derivative for three-dimensional linear elasticity problems, adopting the total potential energy as cost function and the equilibrium equation as constraint. This method, based on classical shape sensitivity analysis, leads to a systematic procedure to calculate the topological derivative. In particular, firstly we present the mechanical model, later we perform the shape derivative of the corresponding cost function and, finally, we calculate the final expression for the topological derivative using the Topological-Shape Sensitivity Method and results from classical asymptotic analysis around spherical cavities. In order to point out the applicability of the topological derivative in the context of topology optimization problems, we use this information as a descent direction to solve a three-dimensional topology design problem. Furthermore, through this example we also show that the topological derivative together with an appropriate mesh refinement strategy are able to capture high quality shapes even using a very simple topology algorithm.

Keywords: Topological derivative; Topological-Shape Sensitivity Method; Shape sensitivity analysis; Topology design

1. Introduction

The topological derivative has been recognized as an alternative methodology and at the same time a promising tool to solve topology optimization problems (see [5,6,10,30] and references therein). Moreover, this is a broad concept. In fact, the topological derivative may also be applied to analyze any kind of sensitivity problem in which discontinuous changes are allowable, for example, discontinuous changes on the shape of the boundary, on the boundary conditions, on the load system and/or on the parameters of the problem. The information given by the topological derivative is very useful in solving problems such as topology design, inverse problems (domain, boundary conditions and parameters characterization), image processing (enhancement and segmentation) and in the mechanical modeling of problems with changes on the configuration of the domain like fracture mechanics and damage. An extension of topological derivative in order to include arbitrary shaped holes and its applications to Laplace, Poisson, Helmoltz, Navier, Stokes and Navier–Stokes equations were developed by Masmoudi and Sokolowski and their respective co-workers (see, for instance, [2,24] for applications of the topological derivative in the context of topology design and inverse problems).

Although the topological derivative is extremely general, this concept may become restrictive due to mathematical difficulties involved in its calculation. To overcame this difficult authors have put forward different approaches to calculate the topological derivative. In particular, we proposed an alternative method based on classical shape sensitivity analysis (see [3,17,18,28,31,32,34] and references therein). This approach, called Topological-Shape Sensitivity Method, has been applied for us in the following two-dimensional problems:

* Corresponding author.
E-mail address: novotny@lncc.br (A.A. Novotny).

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- Poisson: steady-state heat conduction problem taking into account both homogeneous and non-homogeneous Neumann and Dirichlet and also Robin boundary conditions on the hole [8,26].
- Navier: plane stress and plane strain linear elasticity [9].
- Kirchhoff: thin plate bending problem [27].

Specifically, we considered respectively scalar second-order, vector second-order and scalar forth-order PDE two-dimensional problems. As a natural sequence of our research, in the present paper we apply the Topological-Shape Sensitivity Method to calculate the topological derivative in a vector second-order PDE three-dimensional problem. At this moment, we consider the three-dimensional linear elasticity problem taking the total potential energy as cost function and the state equation as constraint. Thus, for the sake of completeness, in Section 2 we present a brief description of the Topological-Shape Sensitivity Method. In Section 3 we use this approach to calculate the topological derivative for the problem under consideration: initially we present the mechanical model associated to three-dimensional linear elasticity, further we calculate the shape derivative for this problem adopting the total potential energy as cost function and the weak form of the state equation as constraint and then we obtain the expression for the topological derivative using classical asymptotic analysis around spherical cavities. Finally, in Section 4 we show that the topological derivative is a powerful tool to be applied in topology optimization context by using it as a descent direction to solve a three-dimensional topology design problem, whose result is improved with help of an appropriate adaptive mesh refinement strategy.

2. Topological-Shape Sensitivity Method

Let us consider an open bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$. If the domain Ω is perturbed by introducing a small hole at an arbitrary point $\hat{\mathbf{x}} \in \Omega$, we have a new domain $\Omega_\varepsilon = \Omega - \overline{B}_\varepsilon$, whose boundary is denoted by $\partial\Omega_\varepsilon = \partial\Omega \cup \partial B_\varepsilon$, where $\overline{B}_\varepsilon = B_\varepsilon \cup \partial B_\varepsilon$ is a ball of radius ε centered at point $\hat{\mathbf{x}} \in \Omega$. Therefore, we have the original domain without hole Ω and the new one Ω_ε with a small hole B_ε as shown in Fig. 1.

Thus, considering a cost function ψ defined in both domains, we have the following topological asymptotic expansion [10]:

$$\psi(\Omega_\varepsilon) = \psi(\Omega) + f(\varepsilon)D_{\mathrm{T}}(\hat{\mathbf{x}}) + \mathscr{R}(f(\varepsilon)), \tag{1}$$

where $f(\varepsilon)$ is a monotone function so that $f(\varepsilon) \to 0$ with $\varepsilon \to 0^+$ and $\mathscr{R}(f(\varepsilon))$ contains all higher order terms than $f(\varepsilon)$, that is, it satisfies

$$\mathscr{R}(f(\varepsilon)) : \lim_{\varepsilon\to 0} \frac{\mathscr{R}(f(\varepsilon))}{f(\varepsilon)} = 0. \tag{2}$$

In addition, we can rewrite Eq. (2) and, after taking the limit $\varepsilon \to 0$, $D_{\mathrm{T}}(\hat{\mathbf{x}})$ may be recognized as the well-known topological derivative, that is

$$D_{\mathrm{T}}(\hat{\mathbf{x}}) = \lim_{\varepsilon\to 0} \frac{\psi(\Omega_\varepsilon) - \psi(\Omega)}{f(\varepsilon)}. \tag{3}$$

Recently an alternative procedure to calculate the topological derivative, called Topological-Shape Sensitivity Method, was introduced by the authors [26]. This approach makes use of the whole mathematical framework (and results) developed for shape sensitivity analysis (see, for instance, the pioneering work of Murat and Simon [23]). The main result obtained in [26] may be briefly summarized in the following theorem (see also [8,25]):

Theorem 1. *Let $f(\varepsilon)$ be a function chosen in order to $0 < |D_{\mathrm{T}}(\hat{\mathbf{x}})| < \infty$, then the topological derivative given by Eq. (3) can be written as*

$$D_{\mathrm{T}}(\hat{\mathbf{x}}) = \lim_{\varepsilon\to 0} \frac{1}{f'(\varepsilon)} \frac{\mathrm{d}}{\mathrm{d}\tau}\psi(\Omega_\tau)\Big|_{\tau=0}, \tag{4}$$

where $\tau \in \mathbb{R}^+$ is used to parameterize the domain. That is, for τ small enough, we have

$$\Omega_\tau := \{\mathbf{x}_\tau \in \mathbb{R}^3 : \mathbf{x}_\tau = \mathbf{x} + \tau\mathbf{v}, \mathbf{x} \in \Omega_\varepsilon\}. \tag{5}$$

Therefore, $\mathbf{x}_\tau|_{\tau=0} = \mathbf{x}$ and $\Omega_\tau|_{\tau=0} = \Omega_\varepsilon$. In addition, considering that $\mathbf{n}$ is the outward normal unit vector (see Fig. 1), then we can define the shape change velocity $\mathbf{v}$, which is a smooth vector field in Ω_ε assuming the following values on the boundary $\partial\Omega_\varepsilon$

$$\begin{cases} \mathbf{v} = -\mathbf{n} & \text{on } \partial B_\varepsilon, \\ \mathbf{v} = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{6}$$

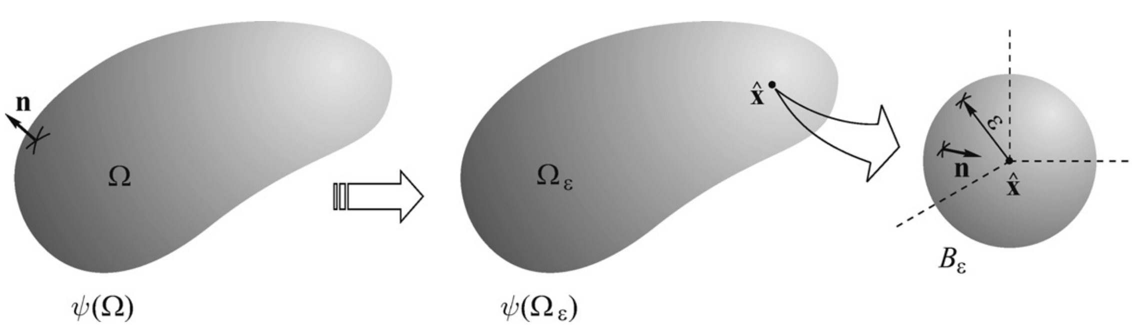

Fig. 1. Topological derivative concept.