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# Inexact Newton-type methods for the solution of steady incompressible viscoplastic flows with the SUPG/PSPG finite element formulation

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**Abstract**

In this work we evaluate the performance of inexact Newton-type schemes to solve the nonlinear equations arising from the SUPG/PSPG finite element formulation of steady viscoplastic incompressible fluid flows. The flow through an abrupt contraction and the rotational flow in eccentric annulus with power law and Bingham rheologies are employed as benchmarks. Our results have shown that inexact schemes are more efficient than traditional Newton-type strategies.


*Keywords:* Inexact Newton; Viscoplastic flow; Stabilized formulations; Finite elements

## 1. Introduction

This work addresses aspects in the finite element simulation of steady viscoplastic flows with emphasis on nonlinear solution strategies employing inexact Newton-type algorithms.

Several modern material and manufacturing processes involve non-Newtonian fluids, and in particular viscoplastic fluids. Examples of non-Newtonian behavior can be found in processes for manufacturing coated sheets, optical fibers, foods, drilling muds and plastic polymers. Numerical simulations of non-Newtonian behavior represent a particular and difficult case of incompressible fluid flows. In these fluids the

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dependence between the viscosity and the shear rate amplifies the nonlinear character of the governing equations. For a comprehensive presentation of numerical methods for non-Newtonian fluid flow computations we refer to [7,16].

The finite element computation of incompressible Newtonian flows involves two sources of potential numerical instabilities associated with the Galerkin formulation of the problem. One source is due to the presence of convective terms in the governing equations. The other source is due to the use of inappropriate combinations of interpolation functions to represent the velocity and pressure fields. These instabilities are frequently prevented by addition of stabilization terms into the Galerkin formulation. In the context of non-Newtonian fluids, the rheological equations are inherently nonlinear, thus increasing the difficulties to find an efficient solution method.

In this work we consider the stabilized finite element formulation proposed by Tezduyar [19] applied to solve steady viscoplastic incompressible flows. This formulation, originally proposed for Newtonian fluids, allows that equal-order-interpolation velocity–pressure elements are employed, circumventing the Babuska–Brezzi stability condition by introducing two stabilization terms. The first term is the streamline upwind Petrov–Galerkin (SUPG) introduced by Brooks and Hughes [6] and the other one is the pressure stabilizing Petrov Galerkin (PSPG) stabilization proposed initially by Hughes et al. [12] for Stokes flows and generalized by Tezduyar et al. [21] to high Reynolds number flows.

It is known that, when discretized, the incompressible Navier–Stokes equations give rise to a system of nonlinear algebraic equations due the presence of convective terms in the momentum equation. Among several strategies to solve nonlinear problems the Newton's methods is attractive because it converges rapidly from any sufficiently good initial guess [8,13]. However, the implementation of Newton's method requires some considerations: Newton's method requires the solution of linear systems at each stage and exact solutions can be too expensive if the number of unknowns is large. In addition, the computational effort spent to find exact solutions for the linearized systems may not be justified when the nonlinear iterates are far from the solution. Therefore, it seems reasonable to use an iterative method [3], such as BiCGSTAB or GMRES, to solve these linear systems only approximately.

The inexact-Newton methods associated with iterative Krylov solvers have been used to reduce computational efforts related to nonlinearities in many problems of computational fluid dynamics, offering a trade-off between the accuracy and the amount of computational effort spent per iteration. According to Kelley [13] its success depends on several factors, such as: quality of initial Newton step, robustness of Jacobian evaluation and proper forcing function choice. Shadid and co-workers presented in [18] an inexact-Newton method applied to problems involving Newtonian fluids, mass and energy transport, discretized by SUPG/PSPG formulation and equal-order-interpolation elements. Recently, Knoll and Keyes [14] discussed the constituents of a broader class of inexact-Newton methods, the Jacobian-free Newton–Krylov methods. In this work we address only the essentials of the inexact-Newton methods and the interested reader should refer to Knoll and Keyes [14], and references therein, for a more detailed presentation.

Many authors have considered finite element formulations in combination with solution algorithms for nonlinear problems arising from non-Newtonian incompressible flow simulations. For instance, in [5] the least-squares method was employed; in [1] a mixed-Galerkin finite element formulation with a Newton–Raphson iteration procedure coupled to an iterative solver was used, while in [17] the authors adopted the Galerkin/least-squares formulation (GLS) associated also with Newton–Raphson algorithm; Meuric et al. in [15] used the SUPG formulation in combination with Newton–Raphson and Picard iterations as a strategy to circumvent computational difficulties in annuli flow computations. Some of these strategies employ analytical or directional forms of Jacobians in the Newton method. The analytical derivative of the stabilization terms are often difficult to evaluate. In this work we have tested the performance of the approximate Jacobian form described by Tezduyar in [20]. This numerically approximated Jacobian is based on Taylor's expansions of the nonlinear terms and presents an alternative and simple way to implement the approximate tangent matrix employed by inexact Newton-type methods.