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Residual-based stabilized higher-order FEM for advection-dominated problems

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Abstract

We reconsider the numerical solution of linear(ized) advection–diffusion–reaction problems using higher-order finite elements together with stabilized Galerkin methods of streamline-diffusion type (SUPG) and with shock-capturing stabilization. The analysis improves the a priori analysis in our previous paper [T. Knopp, G. Lube, G. Rapin, Stabilized finite element methods with shock capturing for advection–diffusion problems, Comput. Methods Appl. Mech. Engrg. 191 (2002) 2997–3013]. The theoretical results are supported by some numerical experiments.

Keywords: Advection–diffusion; Incompressible flow; Stabilized finite elements; SUPG

1. Introduction

The motivation of the present paper stems, e.g., from the finite element simulation of the non-isothermal and incompressible Navier–Stokes problem

$$\partial_t \vec{u} - \nabla \cdot (\nu \nabla \vec{u}) + (\vec{u} \cdot \nabla)\vec{u} + \nabla p = -\beta\theta\vec{g}, \tag{1}$$

$$\nabla \cdot \vec{u} = 0, \tag{2}$$

$$\partial_t \theta + (\vec{u} \cdot \nabla)\theta - \nabla \cdot (a\nabla\theta) = \dot{q}^V / c_p \tag{3}$$

for velocity $\vec{u}$, pressure p and temperature θ in a polyhedral domain $\Omega \subset \mathbf{R}^d$, $d \leqslant 3$, with source terms $\beta\theta\vec{g}$ and $\dot{q}^V/c_p$. This model describes, e.g., the air flow in buildings, etc. [17]. The momentum and continuity Eqs. (1) and (2) describe the fluid motion; the heat transfer is driven by the advection–diffusion Eq. (3).

Turbulence may occur at high Rayleigh or Reynolds numbers. A standard approach is to consider the Reynolds averaged Navier–Stokes equations (RANS) together with, e.g., the k–ϵ turbulence model. Within a statistical turbulence model only averaged values are considered. An eddy viscosity ansatz for turbulent effects is modeled as an additional diffusion term with eddy viscosity ν_t. Then the averaged values for $\vec{u}$, p and θ are determined by (1)–(3) with ν and a replaced with (variable) viscosities $\nu_e = \nu + \nu_t$ and $a_e = a + \nu_t/Pr_t$. The eddy viscosity term ν_t is determined, e.g. in the k–ϵ turbulence model, by $\nu_t = c_\mu k^2/\epsilon$ where the turbulent kinetic energy k and the dissipation rate ϵ of k are defined by additional (nonlinear) advection–diffusion–reaction equations.

A standard algorithmic treatment of the coupled model is to semi-discretize, in an outer loop, in time (with possible step control) using an A-stable method and then, in an inner loop, to decouple and linearize the resulting system. A block

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Gauss–Seidel method with fixed point or Newton-type iteration per time step leads to linearized Navier–Stokes problems (of Oseen-type) and *linearized advection–diffusion–reaction problems* as auxiliary problems, see [17]. A proper numerical approach to the latter model

$$Lu := -\nabla \cdot (a\nabla u) + \vec{b} \cdot \nabla u + cu = f \quad \text{in } \Omega \tag{4}$$

is an important ingredient of the approach and will be discussed in this paper.

The streamline upwind Petrov–Galerkin (SUPG) method, proposed by Brooks and Hughes [1], was the first variationally consistent, stable and accurate finite element model for advection-dominated problems. It initiated the development of stabilization techniques for advection-dominated and related problems. For an overview, see, e.g., [14]. A first relevant analysis of the SUPG method can be found in Johnson et al. [15] in case of regular solutions. The theory has been refined over the years in several directions. Here we only mention the analysis of a *hp*-version by Houston and Süli [12].

Nevertheless, for non-smooth solutions, localized oscillations of the SUPG solution may still exist in the neighborhood of steep gradients. As a remedy, discontinuity- or shock-capturing terms can be added to enhance the stability. Linear (but non-consistent) schemes for low-order elements are considered, e.g., in [16,22]. Mizukami/Hughes [13] introduced the first nonlinear discontinuity-capturing schemes DC1 and DC2. The idea was to enhance, additionally to streamline upwinding, numerical viscosity in the direction of ∇u_h. The consistent approximate upwind (CAU) in [9] provided a further refinement. For recent developments of the CAU scheme to higher-order elements, we refer to [10]. Moreover, Codina considered in [5,6] the discontinuity-capturing/crosswind-dissipation (DC/CD) scheme with additional anisotropic viscosity. The important question of low-order (nonlinear) schemes which satisfy a discrete maximum principle is discussed in the recent papers [2,3], see also the monograph [8].

A first theoretical result for such nonlinear schemes is seemingly due to Szepessy [23]. In our previous paper [18] we considered the a priori analysis of a rather general class of shock-capturing schemes. The goal of the present paper is an extension of the stabilized higher-order FE method of the recent paper [12] to the case of shock-capturing stabilization. In particular, we address the choice of the stabilization parameters, see Section 2, extend and refine the analysis of shock-capturing schemes given in [18], see Section 3, and provide some numerical experiments, see Section 4.

2. Stabilized FEM for advection–diffusion–reaction model

Following basically [12], we describe and analyze the SUPG-stabilization of the advection–diffusion–reaction model. In contrast to [12], we give a refined definition of the stabilization parameters depending on all critical parameters.

2.1. Problem statement

For the advection–diffusion–reaction scheme (4), we assume $a, c \in L^\infty(\Omega)$, $\vec{b} \in (H^1(\Omega))^d \cap (L^\infty(\Omega))^d$, $f \in L^2(\Omega)$ and

$$(\nabla \cdot \vec{b})(x) = 0, \quad c(x) \geqslant \omega \geqslant 0, \quad a(x) \geqslant a_0 > 0, \quad \text{a.e. in } \Omega. \tag{5}$$

For simplicity only, we analyze the homogeneous Dirichlet problem

$$u = 0 \quad \text{on } \partial\Omega. \tag{6}$$

The basic variational formulation of (4)–(6) reads:

$$\text{Find } u \in V := H_0^1(\Omega) \quad \text{s.t. } A(u, v) = l(v) \qquad \forall v \in V \tag{7}$$

with

$$A(u, v) = (a\nabla u, \nabla v)_\Omega + (\vec{b} \cdot \nabla u + cu, v)_\Omega, \tag{8}$$

$$l(v) = (f, v)_\Omega. \tag{9}$$

2.2. Finite element discretization

Suppose a family of admissible triangulations $\mathscr{T}_h = \{T\}$ of the polyhedral domain Ω where h is the piecewise constant mesh function with $h(x) = h_T = \text{diam}(T)$, $x \in T$. We assume that $\mathscr{T}_h$ is shape-regular, i.e. there exists a constant $C_r \neq C_r(h)$ such that

$$C_r h_T^d \leqslant \text{meas}(T) \quad \forall T \in \cup_h \mathscr{T}_h. \tag{10}$$

Moreover, we assume that each element $T \in \mathscr{T}_h$ is a smooth bijective image of a given reference element $\widehat{T}$, i.e., $T = F_T(\widehat{T})$ for all $T \in \mathscr{T}_h$. Here, $\widehat{T}$ is the (open) unit simplex or the (open) unit hypercube in $\mathbf{R}^d$.