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Time step restrictions using semi-explicit methods for the incompressible Navier–Stokes equations

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Abstract

The incompressible Navier–Stokes equations are discretized in space by a finite difference method and integrated in time by the method of lines and a semi-explicit method. In each time step a set of systems of linear equations has to be solved. The size of the time steps is restricted by stability and accuracy of the time-stepping scheme, and convergence of the iterative methods for the solution of the systems of equations. The stability is investigated with a linear model equation derived from the Navier–Stokes equations on Cartesian grids. The resolution in space and time is estimated from turbulent flow physics. The convergence of the iterative solvers is discussed with respect to the time steps. The stability constraints obtained from the model equation are compared to results for a semi-explicit integrator of the Navier–Stokes equations with good agreement. The most restrictive bound on the time step is given by accuracy, stability, or convergence depending on the flow conditions and the numerical method.

MSC: 65M06; 65M12

Keywords: Incompressible flow; Iterative solution; Semi-explicit method; Accuracy constraints; Stability

1. Introduction

Direct simulation of the Navier–Stokes equations (DNS) is a computational tool to study turbulent flow. Spectral and pseudospectral methods have been developed for this purpose but they are restricted to simple geometries such as straight channels. For more complex problems a finite difference or finite element method is more suitable. Examples of such methods are found, e.g., in [5,7,32]. DNS calculations are computationally very demanding with long execution times and large memory requirements. One important issue is how the equations are discretized in space. Another question is how to integrate the equations in time. Given the space discretization, the time derivatives are usually approximated by a standard method for ordinary differential equations [21] or a combination of such methods. The scheme may be implicit [12,42], semi-implicit [17,42], semi-explicit [7,24], or explicit [13,41]. Then in each time step there are in general one or more systems of linear equations to solve for the velocity and the pressure. Preferably the systems of equations are solved by iterative methods since they are superior in efficiency and memory requirements for large problems. In this solution strategy, the time step shall be chosen so that

1. The integration is stable.
2. The solution is sufficiently accurate in time.
3. The iterative solvers converge quickly.

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Less computational work is spent in a time interval if the time steps are long, but long time steps may be in conflict with all three requirements above. Ideally the solution algorithm should be such that only accuracy restricts the time step but this is seldom possible for nonlinear problems. In this paper we investigate this matter and derive bounds on the time step imposed by the stability, accuracy, and convergence for semi-explicit methods in general and the integration method with the finite difference space discretization in [7] in particular.

The Navier–Stokes equations for incompressible flow in two dimensions (2D) in the primitive variables are as follows. Let u and v be the velocity components in the x- and y-directions, respectively, p the pressure, and ν the kinematic viscosity. The Reynolds number is defined by $Re = u_b\ell/\nu$ for some characteristic velocity u_b and length scale ℓ. Let $\mathbf{w} = (u, v)^{\mathrm{T}}$. Introduce the nonlinear and linear terms

$$\mathscr{N}(\mathbf{w}) = (\mathbf{w}\cdot\nabla)\mathbf{w}, \quad \mathscr{L}(\mathbf{w}, p) = \nabla p - Re^{-1}\Delta\mathbf{w}.$$

Then the Navier–Stokes equations in 2D are

$$\partial_t\mathbf{w} + \mathscr{N}(\mathbf{w}) + \mathscr{L}(\mathbf{w}, p) = 0, \tag{1}$$

$$\nabla\cdot\mathbf{w} = 0. \tag{2}$$

The space discretizations of $\mathscr{N}$ and $\mathscr{L}$ in (1) and $\nabla\cdot$ in (2) are denoted by $\mathscr{N}_h$ and $\mathscr{L}_h$ and $\nabla_h\cdot$. Suppose that the solution in space $\mathbf{w}^n$ is known at time t^n and that we intend to compute $\mathbf{w}^{n+1}$ at t^{n+1} with the time step $\Delta t = t^{n+1} - t^n$. In an implicit method, $\mathbf{w}^{n+1}$ fulfills

$$\mathbf{w}^{n+1} + c_1\Delta t\mathscr{N}_h(\mathbf{w}^{n+1}) + c_2\Delta t\mathscr{L}_h(\mathbf{w}^{n+1}, p) = \mathbf{b}^n_{\mathrm{impl}}, \tag{3}$$

where c_1 and c_2 are constants depending on the method and $\mathbf{b}^n_{\mathrm{impl}}$ depends on previous solutions $\mathbf{w}^n$, $\mathbf{w}^{n-1}, \ldots$ In a semi-implicit method the convection term is linearized by introducing a previous solution $\bar{\mathbf{w}}$ in the iterations in the approximation of $\mathscr{N}_h$ such that $\mathscr{N}_h(\mathbf{w}^{n+1}) \approx (\bar{\mathbf{w}}\cdot\nabla)\mathbf{w}^{n+1}$. The nonlinear term is treated explicitly in a semi-explicit method

$$\mathbf{w}^{n+1} + c_1\Delta t\mathscr{L}_h(\mathbf{w}^{n+1}, p) = \mathbf{b}^n_{\mathrm{sexpl}}, \tag{4}$$

where $\mathbf{b}^n_{\mathrm{sexpl}}$ includes $\mathscr{N}_h$ and depends on $\mathbf{w}^n, \mathbf{w}^{n-1}, \ldots$ In an explicit scheme, also the linear term is evaluated from previous solutions

$$\mathbf{w}^{n+1} = \mathbf{b}^n_{\mathrm{expl}}, \tag{5}$$

so that $\mathbf{w}^{n+1}$ is updated without the need to solve a system of equations for the velocity.

The integration method in [7,8,33] is semi-explicit as in (4) and second order accurate with a fourth order accurate compact finite difference discretization of the space derivatives in 2D [9,27]. The solution is expanded in a Fourier series in the third dimension [8]. The system of linear equations for $\mathbf{w}^{n+1}$ and p in each time step is solved in one outer and two inner iterations. The analysis developed here is applied to this method as an example. Although the stability analysis is restricted to this particular space discretization on Cartesian equidistant grids including a linearization, we believe that the trends are more generally applicable to other spatial approximations. The stability constraints are e.g., modified only slightly for a second order method.

There are different options to satisfy the incompressibility condition (2) and to determine the pressure p at t^{n+1}. One possibility is to compute a provisional $\mathbf{w}^*$ and then add a correction so that (2) is satisfied in a pressure correction method or a projection method [4,6,13,15,41,23]. An approximate factorization is determined in a fractional step method to obtain two simpler systems of equations [11,14,26,36]. Another way is to solve (3) or (4) and (2) for $\mathbf{w}^{n+1}$ and p simultaneously and iterate until convergence as in [7,17,43].

The stability and accuracy of semi-explicit (or mixed explicit/implicit) integration methods for the incompressible Navier–Stokes equations are investigated in [24]. The length of the time steps is studied in [12] for turbulent flow with an implicit treatment of $\mathscr{N}_h$ and $\mathscr{L}_h$. A discussion of appropriate time steps for accuracy and stability in turbulent flow is found in [19]. The stability and accuracy of combinations of implicit and explicit methods for one-dimensional, scalar convection–diffusion equations are evaluated in [3].

The time and space discretizations are discussed in Section 2. The time derivative is approximated either by a backward differentiation formula (BDF) or an Adams method. The nonlinear convection term is extrapolated from old solutions or advanced by an explicit Adams method. The analysis of the stability of the discretization in 2D is based on the Oseen equations with frozen velocity coefficients in the nonlinear term $\mathscr{N}_h$ in Section 3. Stability of the particular semi-explicit scheme in [7] is studied in [20] using Fourier analysis. This analysis is generalized here to other classes of semi-explicit methods. If stability problems are indicated using Fourier analysis with locally frozen coefficients, then such trouble is likely to occur also in the nonlinear problem. The model for the stability analysis is validated by comparison of the predictions with the results from calculations with the Navier–Stokes solver [7] in a straight channel. The methodology is easily applicable to other space discretizations. The maximum lengths of the temporal and spatial steps for sufficient accuracy are estimated in