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Maximum principle for mean-field zero-sum stochastic differential game with partial information and its application to finance

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ABSTRACT

In this paper, we investigate the stochastic optimal control problem for the zero-sum stochastic differential game of mean-field type with partial information. We derive a necessary and sufficient maximum principle for that problem by virtue of the duality method and the mean-field backward stochastic differential equations. As an application, we apply the result to the mean-field stochastic differential portfolio game problem, and obtain an equilibrium point of such game.

1. Introduction

Recently, there has been an increasing interest in mean-field models due to the fact that modeling collective behaviors of individuals on account of their mutual interactions in various physical or biological or sociological dynamical systems has been one of the major problems in the history of mankind. It is very convenient to construct the mean field theory to describe such interacting particle systems. The concept of mean field is from statistical physics and the novelty of this theory is that particles interact through a medium, namely the mean field term, aggregated by action of and reaction on each particle. Several examples may be referred to [4].

Over the past few decades, game theory has been an active area of research in operations research and control theory. In addition, game theory is a useful tool in many applications, particularly in biology and financial economic. Mataramvura and Øksendal [19] solved a stochastic differential game (SDG) with the restriction to consider only Markov controls. So the equilibrium point is derived using the Hamilton–Jacobi–sBellman (HJB) equations. Later, An and Øksendal [2] studied the SDGs with partial information. They established a maximum principle for such stochastic control problems. Chen and Yu [11] gave a maximum principle for nonzero-sum stochastic differential game with delays. More recently, mean-field game theory has raised much popular interest since the independent introduction by Huang–Caines–Malhamé (where the framework was called the Nash Certainty Equivalence Principle) and Lasry–Lions, see Huang et al. [14] and Lasry and Lions [16] where the now standard terminology of Mean Field Games was introduced. In fact, the mean field equations for dynamical games with large but finite populations of asymptotically negligible agents originated in the work of Huang et al. [15]. Over recent years it has been a rapid growth in the literature on mean-field games. The reader can refer to the book by Bensoussan et al. [5] for a review of main results and methods. Specifically, Tembine et al. [25] studied a class of risk-sensitive mean-field stochastic differential games. Nourian and Caines [21] considered large population dynamic games involving nonlinear stochastic dynamical systems with agents. Gomes and Pimentel [13] proved the existence of classical solutions for time-dependent mean-field games with a logarithmic nonlinearity and subquadratic Hamiltonians. Djehiche and Huang [12] discussed a class of dynamic decision problems of mean-field type with time-inconsistent cost functionals and derive a stochastic maximum principle to characterize sub-game perfect equilibrium points. Not long ago, Bensoussan et al. [4] provided a comprehensive study of a general class of linear-quadratic mean-field games. Swiecicki et al. [24] studied a particular class of mean-field games that shows strong analogies with the nonlinear Schrödinger and Gross-Pitaevskii equations introduced in physics to describe a variety of physical phenomena. Ahuja [1] showed the existence and uniqueness of a mean field game solution using the stochastic maximum principle.

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In our mean-field game model, since the control processes are adapted to a given subfiltration of the filtration generated by the underlying Lévy processes and the performance functional is a nonlinear function of the expected value (of mean-field type), this leads to a so-called time inconsistent control problem. Thus, we cannot use dynamic programming and HJB equations to solve the problem. In this paper, we derive sufficient conditions and necessary conditions for optimality of this control problem in the form of stochastic maximum principle. There is already a lot of literature on the maximum principle for the optimal control of mean-field systems. Interested readers may refer to Andersson and Djehiche [3], Buckdahn et al. [8], Li [17], Meyer-Brandis et al. [20], Shen and Siu [22], Shen et al. [23], Wang et al. [26] and Ma and Liu [18] for various versions of the stochastic maximum principles for the mean-field models.

The rest of the paper is organized as follows. In Section 2, we introduce some notations and the formulation of the mean-field zero-sum SDG with partial information. In Section 3, we establish a sufficient maximum principle for the mean-field game problem. Section 4 is devoted to derive a necessary maximum principle for the mean-field game problem. An application to the mean-field stochastic differential portfolio game problem is presented in Section 5. Finally, Section 6 concludes the paper.

2. Formulation of the game

Let $\mathcal{T} := [0, T]$ denote a finite horizon, where $T < \infty$. Let $B(t) = (B_1(t), \dots, B_d(t))^T$ (where $()^T$ denotes transposed) and $\eta(t) = (\eta_1(t), \dots, \eta_l(t))^T$ be d−dimensional standard Brownian motion and l independent pure jump Lévy martingales, respectively, on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Let $\tilde{N}(dt, dz) = (\tilde{N}_1(dt, dz_1), \dots, \tilde{N}_l(dt, dz_l))^T$, where $\tilde{N}_i(dt, dz_i) := N_i(dt, dz_i) - \nu_i(dz_i)dt$ is the compensated jump measure of $\eta_i(\cdot)$, $1 \leq i \leq l$. Here, $\nu_i(dz_i)$ is the Lévy measure of $\eta_i(\cdot)$ and $N_i(dt, dz_i)$ is the jump measure of $\eta_i(\cdot)$. We assume that the Brownian motion and the pure jump Lévy martingales are stochastically independent under $\mathbb{P}$ and $\mathbb{F} := \{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by B and $\tilde{N}$ (augmented with the $\mathbb{P}$-null sets of $\mathcal{F}$). We can write

$$\eta_i(t) = \int_0^t \int_{\mathbb{R}_0} z_i \tilde{N}_i(ds, dz_i), \quad i = 1, \dots, l,$$

where $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$.

We assume that the dynamics of a stochastic system is modeled by the following mean-field jump-diffusion stochastic differential equation (SDE) of the form

$$\begin{cases} dX^{u_1,u_2}(t) = \mu(\Xi_1^{u_1,u_2}(t))dt + \sigma(\Xi_2^{u_1,u_2}(t))dB(t) \\ \qquad\qquad + \int_{\mathbb{R}_0^l} \gamma(\Xi_3^{u_1,u_2}(t,z))\tilde{N}(dt,dz), t \in \mathcal{T}, \\ X^{u_1,u_2}(0) = x, \end{cases} \tag{2.1}$$

where

$$\Xi_1^{u_1,u_2}(t) := (t, X^{u_1,u_2}(t), \mathbb{E}[\phi(X^{u_1,u_2}(t))], u_1(t), u_2(t)),$$

$$\Xi_2^{u_1,u_2}(t) := (t, X^{u_1,u_2}(t), \mathbb{E}[\varphi(X^{u_1,u_2}(t))], u_1(t), u_2(t)),$$

$$\Xi_3^{u_1,u_2}(t,z) := (t, X^{u_1,u_2}(t), \mathbb{E}[\psi(X^{u_1,u_2}(t))], u_1(t), u_2(t), z).$$

Here $\mathbb{E} = \mathbb{E}_{\mathbb{P}}$ denotes expectation with respect to $\mathbb{P}$ and $\mu : \Omega \times \mathcal{T} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbf{U}_1 \times \mathbf{U}_2 \to \mathbb{R}^n$, $\sigma : \Omega \times \mathcal{T} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbf{U}_1 \times \mathbf{U}_2 \to \mathbb{R}^{n\times d}$, $\gamma : \Omega \times \mathcal{T} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbf{U}_1 \times \mathbf{U}_2 \times \mathbb{R}^l \to \mathbb{R}^{n\times l}$ are $\mathbb{F}$-progressively measurable processes, and $\phi : \mathbb{R}^n \to \mathbb{R}^n$, $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, $\psi : \mathbb{R}^n \to \mathbb{R}^n$, are given functions. The control domains $\mathbf{U}_1$ and $\mathbf{U}_2$ are two nonempty convex subsets of $\mathbb{R}^{k_1}$ and $\mathbb{R}^{k_2}$, respectively. $u_1(\cdot)$ and $u_2(\cdot)$ are control processes of Player 1 and Player 2, respectively. We require that the control processes $u_1(\cdot)$ and $u_2(\cdot)$ are càdlàg and adapted to a given filtration $\mathbb{G} := \{\mathcal{E}_t\}_{t\geq 0}$. Here, $\mathcal{E}_t$ is a sub-sigma algebra of $\mathcal{F}_t$ at time t, i.e., $\mathcal{E}_t \subseteq \mathcal{F}_t$, $t \geq 0$. The sub-sigma algebra is very general. For instance, we could have $\mathcal{E}_t = \mathcal{F}_{(t-\delta)^+}$, $t \geq 0$, where $(t-\delta)^+ = \max\{0, t-\delta\}$. This represents that the players only have delayed information available about the state of the system. This is so called partial information. We emphasize that our partial information must be distinguished from partial observation. The mean-field SDE (2.1) which is also called McKean–Vlasov-type equation is obtained as the mean square limit of an interacting particle system of the form (when $n \to \infty$)

$$\begin{aligned} dX^{i,n}(t) &= \mu\left(t, X^{i,n}(t), \frac{1}{n}\sum_{i=1}^n \phi(X^{i,n}(t)), u_1(t), u_2(t)\right)dt \\ &+ \sigma\left(t, X^{i,n}(t), \frac{1}{n}\sum_{i=1}^n \varphi(X^{i,n}(t)), u_1(t), u_2(t)\right)dB^i(t) \\ &+ \int_{\mathbb{R}_0^l} \gamma\left(t, X^{i,n}(t), \frac{1}{n}\sum_{i=1}^n \psi(X^{i,n}(t)), u_1(t), u_2(t), z\right)\tilde{N}^i(dt, dz). \end{aligned}$$

Let

$$\xi_1 := (x, \phi, u_1, u_2), \quad \xi_2 := (x, \varphi, u_1, u_2), \quad \xi_3 := (x, \psi, u_1, u_2),$$

where $\phi = \mathbb{E}[\phi(x)]$, $\varphi = \mathbb{E}[\varphi(x)]$, $\psi = \mathbb{E}[\psi(x)]$, $x, y, z \in \mathbb{R}^n$, $u_1 \in U_1$ and $u_2 \in U_2$. We denote the norm in $\mathbb{R}^n$ by $|\cdot|$. We will also use the following notations ($p \geq 1$):

$$L^p(\mathcal{F}_T; \mathbb{R}^n) = \{\xi : \xi \text{ is } \mathbb{R}^n - \text{valued } \mathcal{F}_T - \text{measurable random variable such that } \mathbb{E}[|\xi|^p] < \infty\},$$

$$L^p_{\mathbb{F}}(\mathcal{T}; \mathbb{R}^n) = \left\{Y(t) : \{Y(t), t \in \mathcal{T}\} \text{ is } \mathbb{R}^n - \text{valued } \mathbb{F} - \text{adapted process such that } \mathbb{E}\left[\int_0^T |Y(t)|^p dt\right] < \infty\right\},$$

$$L^2_\nu(\mathcal{T}; \mathbb{R}^{n\times l}) = \{Y(t,z) : \{Y(t,z), (t,z) \in \mathcal{T} \times \mathbb{R}_0^l\} \text{ is } \mathbb{R}^{n\times l} - \text{valued } \mathcal{P} \otimes \mathcal{B}(\mathbb{R}_0^l) - \text{adapted process such that } \mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0^l} \text{tr}[Y(t,z)^T Y(t,z)\text{diag}(\nu(dz))]dt\right] < \infty,$$

where $\mathcal{P}$ denotes the σ − field of $\mathbb{F}$ − predictable sets on $\Omega \times \mathcal{T}\}$,

$\mathcal{S}^2(\mathcal{T}; \mathbb{R}^n) = \{Y(t) : \{Y(t), t \in \mathcal{T}\}$ is $\mathbb{F}$-adapted càdlàg process such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |Y(t)|^2\right] < \infty\bigg\}.$$

Suppose the following conditions are satisfied.

(A1) μ, σ and γ are uniformly Lipschitz and have a linear growth in ξ_1, ξ_2 and ξ_3, respectively. ϕ, φ and ψ are uniformly Lipschitz and have a linear growth;

(A2) For any $\xi_i \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbf{U}_1 \times \mathbf{U}_2$, stochastic processes $\mu(\cdot, \xi_1) \in L^2_{\mathbb{F}}(\mathcal{T}; \mathbb{R}^n)$, $\sigma(\cdot, \xi_2) \in L^2_{\mathbb{F}}(\mathcal{T}; \mathbb{R}^{n\times d})$ and $\gamma(\cdot, \xi_3, \cdot) \in L^2_\nu(\mathcal{T}; \mathbb{R}^{n\times l})$.

Definition 1. The control processes $u_1(\cdot)$ and $u_2(\cdot)$ are called to be admissible if $u_1(\cdot) \in L^2_{\mathbb{G}}(\mathcal{T}; \mathbf{U}_1)$ and $u_2(\cdot) \in L^2_{\mathbb{G}}(\mathcal{T}; \mathbf{U}_2)$. The process $u_i(\cdot) \in L^2_{\mathbb{G}}(\mathcal{T}; \mathbf{U}_i)$ is called an admissible control of Player i