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Rigorous constraint satisfaction for sampled linear systems

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ABSTRACT

We address a specific but recurring problem related to sampled linear systems. In particular, we provide a numerical method for the rigorous verification of constraint satisfaction for linear continuous-time systems between sampling instances. The proposed algorithm combines elements of classical branch and bound schemes from global optimization with a recently published procedure to bound the exponential of interval matrices.

1. Introduction and problem statement

We consider continuous-time linear systems

$$\dot{x}(t) = A\,x(t) + B\,u(t), \quad x(0) = x_0 \tag{1}$$

with state and input constraints of the form

$$x(t) \in \mathcal{X} \quad \text{and} \quad u(t) \in \mathcal{U} \quad \text{for every } t \in \mathbb{R}_0 \tag{2}$$

under piecewise constant control

$$u(t) = u(t_k) \quad \text{for every } t \in [k\,\Delta t, (k+1)\Delta t), \tag{3}$$

where $\Delta t > 0$ denotes the sampling time and where $t_k := k\,\Delta t$ for every $k \in \mathbb{N}$. The sets $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{U} \subset \mathbb{R}^m$ are assumed to be convex and compact polytopes containing the origin as an interior point. During controller design (and controller evaluation), system (1) is usually replaced by the discrete-time system

$$x(t_{k+1}) = \widehat{A}\,x(t_k) + \widehat{B}\,u(t_k), \quad x(0) = x_0 \tag{4}$$

with $\widehat{A} := \exp(A\,\Delta t)$ and $\widehat{B} := \int_0^{\Delta t} \exp(A\,\tau)\,\mathrm{d}\tau\,B$. While the discretized system and the continuous-time system coincide at all sampling instances, it is well-known that the continuous-time trajectory may violate the state constraints even though the discrete-time counterpart does not (see, e.g., the motivating example in [15]). This problem can be prevented by considering adapted constraints for the discretized system such that constraint satisfaction of (4) w.r.t. the adapted constraints implies constraints satisfaction of (1) w.r.t. the original constraints (2). Suitable methods for the computation of adapted constraints can, for example, be found in [1,2,9,13,15].

Comparing the methods in [1,2,9,13,15], it is peculiar that the procedures in [1,2,13] all rely on a similar non-convex optimization problem (OP). In fact, the central element of [1, Theorem 5], [2, Eq. (15.16)], and [13, Eq. (15)] is an OP, which can be characterized as follows. For a finite number of tuples $(x_0, u_0) \in \mathcal{X} \times \mathcal{U}$ that satisfy $\widehat{A}\,x_0 + \widehat{B}u_0 \in \mathcal{X}$ (i.e., the successor of the discretized system satisfies the state constraints), we have to guarantee that the associated trajectory of the continuous-time system does not violate the state constraints for any $t \in (0, \Delta t)$. Having this guarantee for a single trajectory is not very meaningful. However, guaranteeing constraint satisfaction for, say, $s \in \mathbb{N}$ tuples $(x_i, u_i) \in \mathcal{X} \times \mathcal{U}$ implies that the continuous-time trajectory associated with any initial condition $(x_0, u_0) \in \mathrm{conv}\{(x_1, u_1), \ldots, (x_s, u_s)\}$ does not violate the original constraints (see [13, Proposition 2]) for details). The computation of adapted constraints for the discretized system (4) can thus be reduced to the analysis of a finite number of continuous-time trajectories (see [1,2,13]).

The problem of guaranteeing constraint satisfaction of the continuous-time trajectory associated with a given tuple $(x_0, u_0) \in \mathcal{X} \times \mathcal{U}$ can be described more precisely along the following lines. First note that the polytope $\mathcal{X}$ can be written in the form

$$\mathcal{X} = \{x \in \mathbb{R}^n \mid Hx \le \mathbf{1}\},$$

where $H \in \mathbb{R}^{p \times n}$ and where $\mathbf{1} \in \mathbb{R}^p$ is a vector with all entries equal to 1. Now, let $\varphi(t, x_0, u_0)$ denote the solution of (1) at time $t \in [0, \Delta t]$ for an initial condition $x_0 \in \mathcal{X}$ and a control action $u_0 \in \mathcal{U}$. Then, the trajectory of the continuous-time system does obviously not

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violate the state constraints for any $t \in [0, \Delta t]$ if

$$\max_{j \in \{1,\dots,p\}} \max_{t \in [0,\Delta t]} e_j^T H\, \varphi(t, x_0, u_0) \le 1, \tag{5}$$

where $e_j \in \mathbb{R}^p$ is the j-th Euclidean unit vector. Taking into account that $\varphi(t, x_0, u_0)$ reads

$$\varphi(t, x_0, u_0) = \exp(A\, t)\, x_0 + \int_0^t \exp(A\, \tau)\, \mathrm{d}\tau\, B\, u_0 \tag{6}$$

for every $t \in [0, \Delta t]$, it is easy to see that $e_j^T H \varphi(t, x_0, u_0)$ is, in general, not concave (nor convex) in t. Hence, verifying whether (5) holds (or not) is a multivariate non-convex OP. Fortunately, the l.h.s. in (5) can be easily decomposed into p univariate OPs of the form

$$f^* := \max_{t \in [0,\Delta t]} f(t), \tag{7}$$

where $f : [0, \Delta t] \to \mathbb{R}$ is given by

$$f(t) := h^T \left(\exp(A\, t)\, x_0 + \int_0^t \exp(A\, \tau)\, \mathrm{d}\tau\, B\, u_0 \right) \tag{8}$$

with $h \in \mathbb{R}^n$. Clearly, (5) holds if $f^* \le 1$ results from (7) for every $h \in \{H^T e_1, \dots, H^T e_P\} \subset \mathbb{R}^n$.

As indicated above, the solution of the non-convex OP (7) for different $(x_0, u_0) \in \mathcal{X} \times \mathcal{U}$ and different $h \in \mathbb{R}^n$ is essential for the methods introduced in [1,2,13]. However, the authors of [1,2,13] do not spend much effort on an efficient solution of (7). In fact, they argue that, although the OP (7) is generally non-convex, it can be solved reliable (using *local* optimization solvers) since it is the search of the maximum of a scalar function on a scalar compact domain. While this observation is true, we can provide more elaborated solution strategies for (7) based on the special structure of the objective function in (8). In this paper, we thus address the rigorous (or *global*) solution of (7) using interval arithmetic (IA, see [7,11] for an overview). More precisely, we intend to identify non-decreasing, non-increasing, convex, and concave segments of $f(t)$ on $[0, \Delta t]$ based on interval inclusions for the first and second time-derivative of $f(t)$. Clearly, for such segments, local maxima can be easily computed and subsequently finding the global maximum is straightforward. The proposed solution scheme for (7) can be readily integrated into the methods in [1,2,13] and thus improves these procedures for the computation of adapted constraints.

The paper is organized as follows. We state basic notation and preliminaries in Section 2. The main result of the paper, i.e., a tailored branch and bound algorithm for the rigorous solution of (7) is presented in Section 3. Finally, the proposed method is illustrated with some examples in Section 4 before giving conclusions in Section 5.

2. Notation and preliminaries

As mentioned in the introduction, we exploit IA to provide interval inclusions for $f(t)$ and its derivatives

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} := f'(t) \quad \text{and} \quad \frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2} := f''(t).$$

IA can be understood as the extension of operations associated with real numbers, like addition or multiplication, to intervals (see, e.g., [11, Section 2.2]). In this paper, we only require a few interval operations summarized in the following lemma.

Lemma 1 (*[11, Eqs. (2.14) and (2.19)]*). *Let $[c] = [\underline{c}, \overline{c}] \subset \mathbb{R}$ and $[d] = [\underline{d}, \overline{d}] \subset \mathbb{R}$ be intervals with $\underline{c} \le \overline{c}$ and $\underline{d} \le \overline{d}$. Define the intervals*

$$[c] + [d] := \left[\underline{c} + \underline{d}, \overline{c} + \overline{d} \right] \quad \textit{and}$$

$$[c] \times [d] := \left[\min\{\underline{c}\,\underline{d}, \underline{c}\,\overline{d}, \overline{c}\,\underline{d}, \overline{c}\,\overline{d}\}, \max\left\{\underline{c}\,\underline{d}, \underline{c}\,\overline{d}, \overline{c}\,\underline{d}, \overline{c}\,\overline{d}\right\} \right].$$

Then, $c + d \in [c] + [d]$ and $c\, d \in [c] \times [d]$ for every $c \in [c]$ and every $d \in [d]$.

The rules in Lemma 1 can also be applied to compute the sum (or the multiplication) of an interval $[c]$ and a real number $d \in \mathbb{R}$. In this case, $[d]$ can be construed as a degenerated interval with $\underline{d} = \overline{d} = d$. Moreover, by setting $[d] = [c]$, the interval multiplication can be used to evaluate $[c]$ raised to the power of $\kappa \in \mathbb{N}$. However, tighter inclusions result for the calculation rule given in [11, Eq. (3.10)]. In fact, we find $c^\kappa \in [c]^\kappa$ for every $c \in [c]$, where

$$[c]^\kappa := \begin{cases} [\underline{c}^\kappa, \overline{c}^\kappa] & \text{if } \underline{c} > 0 \text{ or } \kappa \text{ is odd}, \\ [\overline{c}^\kappa, \underline{c}^\kappa] & \text{if } \overline{c} < 0 \text{ and } \kappa \text{ is even}, \\ [0, |[c]|^\kappa] & \text{if } 0 \in [c] \text{ and } \kappa \text{ is even}, \end{cases}$$

and where the magnitude of $[c]$ is defined as $|[c]| := \max\{|\underline{c}|, |\overline{c}|\}$. In addition, we define the width of an interval as $w([c]) := \overline{c} - \underline{c}$. IA can be easily extended to interval vectors and interval matrices. For two interval matrices $[C] = [\underline{C}, \overline{C}]$ and $[D] = [\underline{D}, \overline{D}]$ of appropriate size, the sum $[C] + [D]$ and the multiplication $[C]\,[D]$ are understood component-wise. Analogously, the magnitude $|[C]|$ is defined component-wise, i.e., $(|[C]|)_{ij} := |[\underline{C}_{ij}, \overline{C}_{ij}]|$ .Finally, the infinity norm of an interval matrix is defined as the maximum of the norms of the contained real matrices, i.e., $\|[C]\|_\infty := \max_{C \in [C]} \|C\|_\infty$. It is easy to see, that this definition implies $\|[C]\|_\infty = \||[C]|\|_\infty$. Computing interval inclusions for (8) will mainly build on interval inclusions for matrix exponentials, which can be calculated as follows.

Theorem 2 (*[4, Theorem 4.3]*). *Let $[C] = [\underline{C}, \overline{C}]$ be an interval matrix with $\underline{C}, \overline{C} \in \mathbb{R}^{q \times q}$. Let $k, l \in \mathbb{N}$ be such that $2^l (k+2) > \|[C]\|_\infty$. Define $[C^*] := \frac{1}{2^l}[C]$,*

$$[D^*] := I_q + \frac{[C^*]}{1}\left(I_q + \frac{[C^*]}{2}\left(\dots \left(I_q + \frac{[C^*]}{k} \right) \dots \right) \right) + \frac{\|[C^*]\|_\infty^{k+1}}{(k+1)!\left(1 - \frac{\|[C^*]\|_\infty}{k+2} \right)} [-I_q, I_q],$$

and $[D] := [D^*]^{2^l}$. Then $\exp(C) \in [D]$ for every $C \in [C]$.

Note that there exist many ways to evaluate $[D^*]^{2^l}$ as occurring in Theorem 2. In [4, p. 61], the authors propose to use l successive interval square operations, i.e.,

$$[D^*]^{2^l} = \left(\dots \left([D^*]^2 \right)^2 \dots \right)^2.$$

An efficient procedure for the computation of the square of an interval matrix is presented in [8, Section 6].

3. Rigorous solution via interval arithmetic

In the following, we present a tailored method for the rigorous solution of the non-convex OP (7). Before describing the algorithm, we have to stress that there exists a number of situations where (7) can be solved analytically. In this context, note that (8) can be rewritten as

$$f(t) = h^T \left(\int_0^t \exp(A\, \tau)\, \mathrm{d}\tau (A\, x_0 + B\, u_0) + x_0 \right) \tag{9}$$

using the identity $\int_0^t \exp(A\, \tau)\, \mathrm{d}\tau\, A + I_n = \exp(A\, t)$. Obviously, trivial solutions result if $A\, x_0 + B\, u_0 = 0$, $A = 0$, or $h = 0$. In addition, an analytical solution of (7) is straightforward if h is an eigenvector of A^T, i.e., if $h^T A = \lambda\, h^T$ for some $\lambda \in \mathbb{R}$. To see this, note that the time-derivatives