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$\mathcal{H}_\infty$ Analysis of Linear Systems with Jumps: Applications to Sampled-Data Control ⋆

Matheus Souza * **André R. Fioravanti** ** **Alim P. C. Gonçalves** *

* *School of Electrical and Computer Engineering, University of Campinas, Brazil (e-mail: {msouza,alimped}@dca.fee.unicamp.br)*
** *School of Mechanical Engineering, University of Campinas, Brazil (e-mail: fioravanti@fem.unicamp.br)*

Abstract: This note focuses on $\mathcal{H}_\infty$ analysis of a certain class of hybrid linear systems. The main results presented in this paper are embedded in the context of Riccati equations and convex optimisation. These results, together with the classic Small-Gain Theorem, can be applied to design state feedback controllers for sampled-data systems subject to time-delays.

Keywords: Hybrid systems, Sampled-data systems, $\mathcal{H}_\infty$ control, Linear systems, LMI.

1. INTRODUCTION

Hybrid (or jump) linear systems are dynamic models that combine both continuous and discrete-time behaviours in their formulation (Goebel et al., 2009). Specific classes of hybrid systems that are of great practical interest comprise switched systems (Sun and Ge, 2005) and Markov jump systems (Costa et al., 2013). The reader should refer to the aforementioned references in addition to (Lunze and Lamnabhi-Lagarrigue, 2009; Shorten et al., 2007) for an overview of important results for this class of systems.

Additionally to the previously mentioned classic hybrid models, jump systems also provide a natural time-domain formulation of several sampled-data control and filtering problems. A comprehensive analysis of both jump and sampled-data systems is done in (Ichicawa and Katayama, 2001), where the authors present meaningful results based on boundary value problems and on Riccati equations. There exist several references in the literature to date that extend the basic results provided in (Ichicawa and Katayama, 2001) in a broad range of directions. Important recent developments obtained for such systems are based on convex descriptions of the main stability conditions provided in that reference. In (Briat, 2013), the authors provide stability and state feedback stabilisation results for sampled-data systems. In (Geromel and Souza, 2015), optimal $\mathcal{H}_2$ and $\mathcal{H}_\infty$ state feedback controllers are devised solving convex optimisation problems. In (Souza and Geromel, 2015), stability and $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance necessary and sufficient convex conditions are also analysed. Similar conditions are also devised for switched linear systems in (Briat, 2015). Other important references that approach the sampled-data control design problem from a jump systems viewpoint are (Hara et al., 1994; Chen. and Francis, 1991; Kabamba. and Hara, 1993; Sun et al., 1993).

The main object of study in this paper is the jump linear system whose realisation is given by

$$\mathcal{H} \; : \; \begin{cases} \dot{x}(t) = Ax(t) + Bw(t), \\ y(t) = Cx(t) + Dw(t), \\ x(t_k) = Kx(t_k^-) + Lw_d(k), \end{cases} \tag{1}$$

which is valid for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$. In this paper, we assume the sequence of *jump instants* $(t_k)_{k\in\mathbb{N}}$ is such that, for given $t_0 \in \mathbb{R}$ and $h \in \mathbb{R}_+ \setminus \{0\}$, $t_{k+1} - t_k = h$, $\forall k \in \mathbb{N}$, implying that the system jumps periodically and that Zeno's phenomenon is ruled out. Denoting $\mathbb{T} := [t_0, \infty)$, the continuous-time signals involved in (1) are the state $x : \mathbb{T} \to \mathbb{R}^{n_x}$, the output $y : \mathbb{T} \to \mathbb{R}^{n_y}$ and the input $w : \mathbb{T} \to \mathbb{R}^{n_w}$; $w_d : \mathbb{N} \to \mathbb{R}^{n_w}$ is a discrete-time input that only acts on the jump equation; the matrix sextuple (A, B, C, D, K, L) is composed of real matrices of compatible dimensions. The system is assumed to evolve from a given initial condition $x(t_0^-) \in \mathbb{R}^{n_x}$; it is also possible, however, to consider initial conditions at t_0 and, thus, initial states at t_0^- are particular cases such that $x(t_0) = Kx(t_0^-)$.

Our main goal is to extend the $\mathcal{H}_\infty$ performance measure defined in (Geromel and Souza, 2015; Souza and Geromel, 2015). Based on this result and on the Small-Gain Theorem, we also show how these conditions can be applied to design sampled-data state-feedback controllers subject to time-delays. An example from the literature shall be used to validate (and to point out the main features of) the main theoretical developments of this note.

Notation. For real matrices and vectors, $(^T)$ indicates transpose. For square matrices, $\mathbf{tr}(\cdot)$ denotes the trace function. The sets of natural, real and nonnegative real numbers are indicated by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$, respectively. Usual norms adopted in this paper are denoted as follows; The Euclidean norm of a vector x in $\mathbb{R}^n$ is denoted by

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$\|x\| \triangleq \sqrt{x'x}$; the $\mathcal{L}_2$ and ℓ_2 norms for signals are denoted by $\|\cdot\|_2$. For symmetric matrices, the symbol $(\star)$ denotes each of its symmetric blocks. Finally, $X > 0$ ($X \geq 0$) denotes that the symmetric matrix X is positive definite (positive semidefinite); the set of all (positive definite) symmetric matrices of order n is denoted by ($\mathbb{S}_+^n$) $\mathbb{S}^n$.

2. JUMP SYSTEMS: DYNAMICS

In this section, we present (and extend) classic results concerning the state dynamics and stability of the hybrid linear system $\mathcal{H}$. We shall assume, for now, that $w, w_d \equiv 0$; that is, the system is autonomous.

2.1 State Dynamics

As in any linear system, it suffices to find a *state transition matrix* associated with (1) to completely describe its dynamic behaviour. As stated in (Ichicawa and Katayama, 2001), $\phi : \mathbb{T} \times \mathbb{T} \to \mathbb{R}^{n\times n}$ is a state transition matrix associated with (1) if, and only if, ϕ satisfies

$$\frac{\partial}{\partial t}\phi(t,s) = A\phi(t,s), \quad \forall t \in (t_k, t_{k+1}), k \in \mathbb{N}, \tag{2}$$

$$\phi(t_k, s) = K\phi(t_k^-, s), \quad \forall k \in \mathbb{N}, \tag{3}$$

$$\phi(t,t) = I. \tag{4}$$

In this case, it follows that any state trajectory $x : \mathbb{T} \to \mathbb{R}^{n_x}$ of $\mathcal{H}$ satisfies $x(t) = \phi(t,s)x(s)$ for all $s, t \in \mathbb{T}$. Moreover, other properties such as $\phi(t_k, t_k^-) = K$ and $\phi(t, t_k^-) = \phi(t, t_k)K$, for all $t \geq t_k$, $k \in \mathbb{N}$, can also be verified. In the particular but important case in which the jump instants sequence is such that $t_{k+1} - t_k = h > 0$ for all $k \in \mathbb{N}$, the state transition matrix also satisfies $\phi(t+h, s+h) = \phi(t,s)$, for all $t, s \in \mathbb{T}$; see (Ichicawa and Katayama, 2001).

2.2 Stability

We are now able to provide necessary and sufficient conditions for the state dynamics of $\mathcal{H}$. First, we have to extend an important stability concept that is widely adopted in the LTI case (Rugh, 1996; Ichicawa and Katayama, 2001).

Definition 1. (Exponential Stability). The hybrid linear system $\mathcal{H}$ is said to be *uniformly exponentially stable* if there exist finite positive constants κ and α such that for any $t_0 \in \mathbb{R}$ and $x(t_0) \in \mathbb{R}^{n_x}$, the corresponding solution $x(\cdot)$ satisfies

$$\|x(t)\| \leq \kappa e^{-\alpha(t-t_0)}\|x(t_0)\|, \quad \forall t \in \mathbb{T}. \tag{5}$$

Alternatively, uniform exponential stability can be completely described by the state transition matrix (Rugh, 1996).

Lemma 2. The hybrid linear system $\mathcal{H}$ is uniformly exponentially stable if, and only if, there exist finite positive constants κ and α such that

$$\|\phi(t,s)\| \leq \kappa e^{-\alpha(t-s)} \tag{6}$$

for all $t, s \in \mathbb{T}$ such that $t \geq s$.

There exist several stability conditions for the hybrid linear system $\mathcal{H}$ that is studied in this paper. We are particularly interested, however, in the necessary and sufficient inequality-based conditions presented in (Geromel and Souza, 2015); note that the proof of the necessity – that is, that (ii) implies (i) – is not presented in that reference and is included now.

Theorem 3. Consider the hybrid linear system $\mathcal{H}$, whose realisation is given by (1), and let $S \in \mathbb{S}_+^n$ and $h \in \mathbb{R}_+^\star$ be given. The following statements are equivalent:

(i) For $w \equiv 0$ and $t_{k+1} - t_k = h$, $\forall k \in \mathbb{N}$, $\mathcal{H}$ is uniformly exponentially stable and the bound

$$\int_{t_0}^{\infty} \|y(t)\|^2 \,\mathrm{d}t < x(t_0^-)^{\mathrm{T}} K^{\mathrm{T}} S K x(t_0^-) \tag{7}$$

holds for any initial condition $x(t_0^-) \in \mathbb{R}^n$.

(ii) There exists a solution $X : [0,h] \to \mathbb{S}_+^n$ to the boundary value problem defined by the linear differential equation

$$\dot{X}(t) + A^{\mathrm{T}}X(t) + X(t)A + C^{\mathrm{T}}C = 0 \tag{8}$$

and by the initial $X(0) = S$ and final $X(h) > K^{\mathrm{T}}SK$ conditions.

(iii) The linear matrix inequality

$$A_h^{\mathrm{T}} K^{\mathrm{T}} S K A_h - S + C_h^{\mathrm{T}} C_h < 0 \tag{9}$$

holds, where the matrix pair (A_h, C_h) is such that

$$A_h = e^{hA}, \quad C_h^{\mathrm{T}} C_h = \int_0^h e^{tA^{\mathrm{T}}} C^{\mathrm{T}} C e^{tA} \,\mathrm{d}t. \tag{10}$$

Proof: We shall prove that (ii) ⇒ (iii) ⇒ (i) ⇒ (ii). To this end, let us define the positive definite quadratic function

$$\begin{aligned} v &: \mathbb{R}_+ \to \mathbb{R}, \\ t &\mapsto x(t)^{\mathrm{T}} P(t) x(t), \end{aligned} \tag{11}$$

in which $P : \mathbb{T} \to \mathbb{S}_+^{n_x}$ is such that $P(t) = X(t - t_k)$ for all $t \in [t_k, t_{k+1})$ and $k \in \mathbb{N}$; such function is a *periodic extension* of X and, thus, $P(t_k) = X(0)$ and $P(t_{k+1}^-) = X(h)$ hold for all $k \in \mathbb{N}$.

First, let us assume that (ii) holds. Since (8) is linear, it is clear that

$$X(t) = e^{-tA^{\mathrm{T}}} X(0) e^{-tA} - \int_0^t e^{(\tau-t)A^{\mathrm{T}}} C^{\mathrm{T}} C e^{(\tau-t)A} \mathrm{d}\tau \tag{12}$$

is its unique solution; see Rugh (1996) for details. In particular, we may take $t = h$ and use both boundary conditions to conclude that the LMI (10) is verified, implying that (iii) holds.

Now, assume that (iii) holds. From (9), there exists $\alpha > 0$ such that

$$e^{hA^{\mathrm{T}}} K^{\mathrm{T}} S K e^{hA} \leq e^{-2\alpha h} S. \tag{13}$$

This inequality implies that

$$v(t_{k+1}) \leq e^{-2\alpha(t_{k+1}-t_k)} v(t_k) \tag{14}$$

holds for all $k \in \mathbb{N}$. Additionally, take any $c_0 \geq 1$ satisfying

$$e^{t(A+\alpha I)^{\mathrm{T}}} X(t) e^{t(A+\alpha I)} \leq c_0 X(0), \quad \forall t \in [0,h), \tag{15}$$

which always exists since the positive definite solution X exists and is bounded. Using (15), we can conclude that

$$\begin{aligned} v(t) &= x(t)^{\mathrm{T}} P(t) x(t) \\ &= x(t_k)^{\mathrm{T}} e^{(t-t_k)A^{\mathrm{T}}} X(t-t_k) e^{(t-t_k)A} x(t_k) \\ &\leq c_0 e^{-2\alpha(t-t_k)} x(t_k)^{\mathrm{T}} P(t_k) x(t_k) \\ &= c_0 e^{-2\alpha(t-t_k)} v(t_k) \end{aligned} \tag{16}$$

holds for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$. Since the boundary conditions also imply that $v(t_k) < v(t_k^-)$ holds for all k,