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Stabilizability of Dynamic Coalitional Games with Transferable Utility ⋆

Michalis Smyrnakis * Dario Bauso * Sandor Veres *

* *Department of Automatic Control and Systems Engineering, The University of Sheffield, Sheffield, UK. (e-mail: m.smyrnakis, d.bauso, s.veres@ sheffield.ac.uk).*

Abstract: Stabilizing allocations in coalitional games with transferable utility are studied within the framework of uncertain and dynamic stochastic systems. We assume that the excess of allocations relative to the evolution of coalition values can be described by a stochastic differential equation. As main contribution we provide a feedback control law of allocations to maintain the coalition within a target set. We show that the resulting dynamics is second moment stable under our feedback linear-saturated control strategy.

Keywords: game theory, agents, coalitional games, cooperative games, robot cooperation.

1. INTRODUCTION

Cooperative game theory provides analytical tools in order to study the behaviour of groups of cooperative players (Osborne , 2004). The most studied cooperative games are coalitional games ("coalition games" for short). Coalition games find applications in many areas such as: communication networks (Saad et al., 2009), smart grids (Saad et al., 2012), reconfigurable robo tics (Ramaekers et al., 2011), swarm robotics (Cheng et al., 2008), multi-robot task allocation (Bayram et.al , 2016).

Following the classification of coalition games in (Saad et al., 2009) we will focus on canonical coalition games with transferable utility (TU). In a canonical game the optimal choice of a player is to form a coalition that includes all the players, namely the grand coalition, which means no players or subset of players benefit from quitting the grand coalition. When this occurs then the *excess*, namely the difference between the allocated amount and the value of the coalition is non-negative.

Within the realm of cooperative game theory, games with transferable utility (TU), are games where the reward of a coalition can be divided in any manner among the players participating in the coalition. The value of a coalition can be interpreted as the reward produced by that coalition, which needs to be distributed among the members of the coalition. The way how this value is distributed is established by so-called allocation rules.

When the values of the coalitions are time-varying and uncertain, and the allocation process occurs continuously in time, then the evolution of the excesses is described by the same differential equation that is used to describe fluid flow systems (Bauso et al., 2010).

In this paper we extend the study of coalition games to the case where the excess evolution is affected by both deterministic and stochastic uncertainties. The resulting dynamics is a stochastic differential equation driven by a Brownian motion. The main contribution of this paper is to prove that such a dynamic system is second-moment stochastically stabilizable. The allocation rule takes the form of a linear-saturated feedback control strategy.

Therefore if the uncertainty of a dynamical task can be modelled as a diffusion process (Avellanda et al., 1995; Wikle , 2003) it is possible to derive the system towards a solution where the grand coalition, i.e. the cooperative behaviour of all players, is the only stable solution of the task. Our analysis is based on a technique similar to the one proposed in (Gomes da Silva , 2001) for polytope bounded controls and disturbances. Stabilizability conditions are obtained via linear matrix inequality (LMI) for polytopic systems (Boyd et al., 1994, Chapter 6).

2. PRELIMINARIES ON TU GAMES

This section overviews coalition games with transferable utility (TU). Let a set $N = \{1, \ldots, n\}$ of players be given, $\mathcal{S}$ be the set of all possible non-empty coalitions, with cardinality $|\mathcal{S}| = 2^N - 1$ and a function $\eta : S \mapsto \mathbb{R}$ defined $\forall S \in \mathcal{S}$ and quantifies the gains of coalition S. We denote by $< N, \eta >$ the TU game with players set N and characteristic function η.

Let us introduce some arbitrary mapping of $\mathcal{S}$ into $M := \{1, \ldots, q\}$ where $q = 2^n - 1$, is the number of non-empty coalitions, namely, the cardinality of $\mathcal{S}$. Denote a generic element of M by j. In other words, the j is an indexing of the elements S_j of $\mathcal{S}$ according to some arbitrary but fixed ordering. Let q be the index of the grand coalition N. We let $\eta_j = \eta(S_j)$ be the value of the characteristic function η associated with a non-empty coalition S_j.

Given a TU game, the main question is how to divide the costs or rewards among the participants of the coalition. A partial answer to the above question lies in the concept of *imputation set*. The imputation set $I(\eta)$ is the set of allocations that are

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- *efficient*, that is, the sum of the components of the allocation vector is equal to the value of the grand coalition, and
- *individually rational*, namely no individual benefits arise by quitting the grand coalition and playing alone.

More formally, the imputation set is a convex polyhedron defined as:

$$I_\eta = \{\tilde{u} \in \mathbb{R}^n | \overbrace{\sum_{i\in N} \tilde{u}_i = \eta_q}^{\text{Efficiency}}, \quad \underbrace{\tilde{u}_i \geq \eta_{\{i\}}, \forall i \in N}_{\text{individual rationality}} \}, \text{ where}$$

$\tilde{u}_i$ is the reward allocated to player i and $\eta_{\{i\}}$ is the gain of the coalition, which consists only from player i.

A stronger solution concept than the imputation set is the *core*. Given any allocation in the core, the players not only do not benefit from quitting the grand coalition and playing alone, but do not benefit by creating any subcoalition either. In this sense the core strengthens the conditions valid for the imputation set. The core is still a polyhedral set which is included in the imputation set.

Definition 2.1. The core of a game $\langle N, v\rangle$ is the set of allocations that satisfy i) efficiency, ii) individual rationality, and iii)stability with respect to subcoalitions: $C(\eta) = \{\tilde{u} \in I(\eta)| \underbrace{\sum_{i\in S_j} \tilde{u}_i \geq \eta_j, \forall S_j \in \mathcal{S}}_{\text{stability w.r.t. subcoalitons}} \}.$

A *dynamic TU game* is described by $< N, \eta(t) >$, where $\eta(t)$ is a time-varying characteristic function representing the values of different coalitions.

Let $B_\mathcal{H}$ be a $(q \times n)$-matrix whose rows are the characteristic vectors $y_j^S \in \mathbb{R}^n$ of each coalition $S_j \in \mathcal{S}$. The characteristic vectors are binary vectors representing the participation or not of a player i in the coalition S_j, where $y_i^{S_j} = 1$ if $i \in S_j$ and $y_i^{S_j} = 0$ if $i \notin S_j$. For any allocation in the *core* of the game $C(\eta(t))$ we have

$$\tilde{u}(t) \in C(\eta(t)) \quad \Leftrightarrow \quad B_\mathcal{H}\tilde{u}(t) \geq \eta(t), \tag{1}$$

where the inequality is to be interpreted component-wise, and for the grand coalition is satisfied with equality due to the efficiency condition of the core, i.e, $\sum_{i=1}^n \tilde{u}_i(t) = \eta_q(t)$, where $\eta_q(t)$ denotes the q_{th} component of $\eta(t)$ and is equal to the grand coalition value.

Let

$$B = \left[B_\mathcal{H} \left| \begin{array}{c} -I \\ ---- \\ 0\dots0 \end{array} \right. \right] \in \{-1, 0, 1\}^{q\times(n+q-1)}. \tag{2}$$

Inequality (1) can be rewritten as an equality by using an augmented allocation vector given by $u := \begin{bmatrix}\tilde{u} \\ s\end{bmatrix} \in \mathbb{R}^{n+q-1}$ where s is a vector of $q-1$ non-negative surplus variables. Then, we have

$$\left[B_\mathcal{H} \left| \begin{array}{c} -I \\ ---- \\ 0\dots0 \end{array} \right. \right] \begin{bmatrix}\tilde{u} \\ s\end{bmatrix} = \begin{bmatrix}\eta_1(t) \\ \vdots \\ \eta_q(t)\end{bmatrix}. \tag{3}$$

Note that each surplus variable s_j corresponds to a coalition S_j of players and describes the difference between the allocated value and the coalitional value, $s_j(t) = \sum_{i\in S_j} \tilde{u}_i(t) - \eta_j(t)$. A positive value for $s_j(t)$ can be interpreted as a debit for the coalition, whereas a negative value can be interpreted as a credit. The main insights is that *if all the surpluses are nonnegative, then the total allocation to any coalition exceeds the value of the coalition itself and the allocation vector lies in the core.* Also, notice that there are only $q-1$ surplus variables because coalition N has no surplus ($\sum_{i\in N} \tilde{u}_i - \eta_q = 0$) due to the efficiency condition of the core. For a 3-player coalitional game equation (3) takes the form

$$\underbrace{\left[\begin{array}{ccc|cccccc} 1&0&0&-1&0&0&0&0&0 \\ 0&1&0&0&-1&0&0&0&0 \\ 0&0&1&0&0&-1&0&0&0 \\ 1&1&0&0&0&0&-1&0&0 \\ 1&0&1&0&0&0&0&-1&0 \\ 0&1&1&0&0&0&0&0&-1 \\ 1&1&1&0&0&0&0&0&0 \end{array}\right]}_{B} \underbrace{\begin{bmatrix}\tilde{u}_1\\ \tilde{u}_2\\ \tilde{u}_3\\ s_1\\ s_2\\ s_3\\ s_4\\ s_5\\ s_6\end{bmatrix}}_{u} = \underbrace{\begin{bmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\\ \eta_5\\ \eta_6\\ \eta_7\end{bmatrix}}_{\eta}$$

3. MODEL AND PROBLEM STATEMENT

We consider an n-player robust dynamical TU game $< N, \eta(t) >$, where $\eta(t)$ is the characteristic function representing the values of different coalitions. We consider the case where the characteristic function can be modelled as a diffusion process with drift, and its evolution is described by the stochastic differential equation:

$$\left.\begin{array}{l} d\eta(t) = w(t)dt - \Sigma d\mathcal{B}(t), \quad \text{in } \mathbb{R}^q, \\ \eta(0) = \eta_0, \end{array}\right\} \tag{4}$$

where $w(t)$ is the drift term, $q = 2^n - 1$ is the number of coalitions, $\Sigma = diag((\sigma_i)_{i=1,\dots,q}) \in \mathbb{R}^{q\times q}$ for given scalars σ_i, full column rank, and $\mathcal{B}_t \in \mathbb{R}^q$ is a q-dimensional Brownian motion, which is independent across its components, independent of the initial state x_0, and independent across time.

The deviation from coalitions' characteristic functions will be denoted by $x(t) \in \mathbb{R}^q$. Then, based on (4) the time evolution of $x(t)$ can be defined through the following stochastic differential equation:

$$\left.\begin{array}{l} dx(t) = -x(t) + (Bu(t) - w(t))dt + \Sigma d\mathcal{B}(t), \\ x(0) = x_0, \end{array}\right\} \tag{5}$$

where matrix B and vector u are defined as in (2).

In essence, every component of vector $Bu(t)$ is the total reward given to the members of a coalition at time t, and from this amount the drift from this reward, $w(t)$, is subtracted. Then, a positive $x(t)$ means positive cumulative excess.

We assume that controls and disturbances are bounded, in the sets $U := \mathbb{R}^{(q-1)+n\times q}$ and $W := \mathbb{R}^q$ respectively. Disturbances are bounded within ellipsoids, i.e.,

$$w(t) \in \mathcal{W} = \{w \in \mathbb{R}^q : w^T R_w w \leq 1\}. \tag{6}$$

The above constraints describe some coupling effect on drift uncertainty. Controls are bounded within polytopes

$$u(t) \in \mathcal{U} = \{u \in \mathbb{R}^{(q-1)+n\times q} : u^- \leq u \leq u^+\} \tag{7}$$

with assigned u^+, u^-. Henceforth, for simplicity we make the assumption $u^+ = -u^-$ which simplifies the tractability, though it is not necessary for validity of the results.