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Optimal Block Length for Data-Rate Minimization in Networked LQG Control

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Abstract: We consider a discrete-time networked LQG control problem in which state information must be transmitted to the controller over a noiseless binary channel using prefix-free codewords. Quantizer, encoder and controller are jointly designed to minimize average data-rate while satisfying required LQG control performance. We study the effects of selecting large block-lengths (data transmission intervals) from the perspectives of information-theoretic advantage due to coding efficiency and control-theoretic disadvantage due to delay. In particular, we demonstrate that the performance of networked control scheme by Tanaka et al. (2016) can be improved by adjusting the block-length optimally. As a by-product of this study, we also show that the data-rate theorem for mean-square stability similar to Nair and Evans (2004) can be recovered by considering sufficiently large block-lengths.

Keywords: LQG control, Quantization, Sampling rates, Huffman codes, Data compression, Stability.

1. INTRODUCTION

In this paper, we consider a discrete-time state-feedback control system in which state information must be transmitted to the controller through a noiseless binary channel at every time step. As depicted in Figure 1, we assume that the plant to be controlled has a linear time-invariant state space model [1]

$$\mathbf{x}_{t+1} = A\mathbf{x}_t + B\mathbf{u}_t + F\mathbf{w}_t \tag{1}$$

where $\mathbf{x}_t$ is the $\mathbb{R}^n$-valued state process, $\mathbf{u}_t$ is the $\mathbb{R}^m$-valued control input, and $\mathbf{w}_t$ is the $\mathbb{R}^p$-valued white Gaussian noise with unit covariance I_p. Assume that $\det(A) \neq 0$ and (A, B) is a stabilizable pair. For every non-negative integer t, let

$$\mathcal{A}_t \subset \{0,1\}^\star \triangleq \{0, 1, 00, 01, 10, 11, 000, \cdots\}$$

be a set of prefix-free variable-length binary codewords (Cover and Thomas, 1991, Ch.5). There are multiple ways to choose a codebook $\mathcal{A}_t$, and we allow the choice to be time-varying. Designing appropriate codebooks $\mathcal{A}_0^\infty = (\mathcal{A}_0, \mathcal{A}_1, \cdots)$ is part of our design problem. We also design an encoder policy

$$\mathcal{E}_0^\infty \triangleq \{e_t(a_t|x_0^t, a_0^{t-1})\}_{t=0,1,\dots} \tag{2}$$

which is a sequence of Borel measurable stochastic kernels [2] on $\mathcal{A}_t$ given $\mathcal{X}_0^t \times \mathcal{A}_0^{t-1}$, and a controller policy

$$\mathcal{D}_0^\infty \triangleq \{d_t(u_t|a_0^t, u_0^{t-1})\}_{t=0,1,\dots} \tag{3}$$

which is a sequence of Borel measurable stochastic kernels on $\mathcal{U}_t$ given $\mathcal{A}_0^t \times \mathcal{U}_0^{t-1}$. Notice that $\mathcal{E}_0^\infty$ and $\mathcal{D}_0^\infty$ are general (possibly non-deterministic) causal decision policies. The length of a codeword $a_t \in \mathcal{A}_t$ transmitted from the encoder to the controller at time step t will be denoted by a random variable $\boldsymbol{\ell}_t$. A triplet $(\mathcal{A}_0^\infty, \mathcal{E}_0^\infty, \mathcal{D}_0^\infty)$ will be called a *design*, and the space of such designs is denoted by Γ.

[1] In this paper, random variables are denoted by lower case bold symbols such as $\mathbf{x}$. Calligraphic symbols such as $\mathcal{X}$ are used to denote sets, and $x \in \mathcal{X}$ is an element. We denote by x_0^t a sequence $x_0, x_1, \dots, x_t$, and $\mathbf{x}_0^t$ and $\mathcal{X}_0^t$ are understood similarly.

[2] Foundational discussions on stochastic kernels can be found in Bertsekas and Shreve (1978).

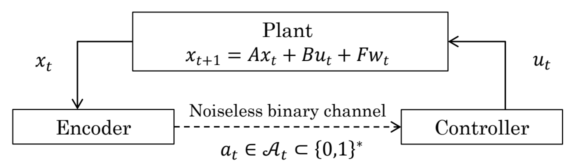

Fig. 1. LQG control over noiseless binary channel.

In this paper, we are interested in a design $(\mathcal{A}_0^\infty, \mathcal{E}_0^\infty, \mathcal{D}_0^\infty)$ that minimizes data rate (i.e., average expected codeword length per time step) while satisfying a required level of LQG control performance γ. Formally, we consider an optimization problem:

$$R(\gamma) \triangleq \min_{\Gamma} \limsup_{T\to+\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}(\boldsymbol{\ell}_t) \tag{4a}$$

$$\text{s.t.} \limsup_{T\to+\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}c(\mathbf{x}_t, \mathbf{u}_t) \leq \gamma. \tag{4b}$$

The cost function $c(\mathbf{x}_t, \mathbf{u}_t)$ is given by a positive definite quadratic form

$$c(\mathbf{x}_t, \mathbf{u}_t) = \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix}.$$

In the optimization problem (4), we require the *average* data rate to be minimized. However, the number of bits $\boldsymbol{\ell}_t$ to be transmitted in a particular time step t can be arbitrary large. Problem (4) is motivated by a common engineering situation in which a digital communication channel must be shared by multiple control systems. For instance, many real-time applications share a common communication bus in an automobile system (e.g., Johansson et al. (2005)). Even though the communication requirements by individual applications might be small, they could collectively cause a serious packet congestion in the shared bus. It is therefore important for individual control applications to minimize their channel use in order to utilize a shared communication resource efficiently.

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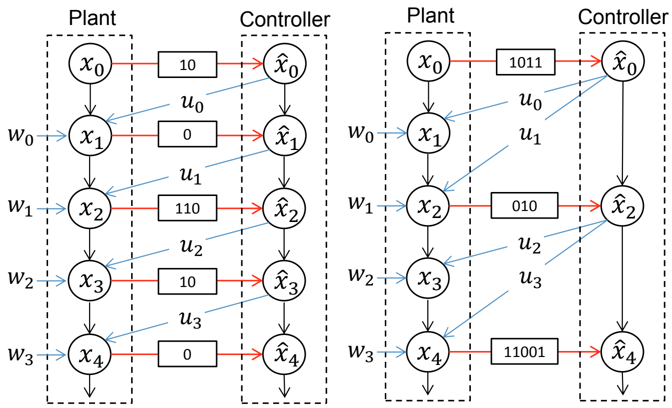

Fig. 2. Standard design (left) and design with block-length two (right).

Obtaining the exact solution to (4) is computationally challenging. However, it is shown by Silva et al. (2011) that a lower bound $DI(\gamma) \leq R(\gamma)$ is obtained by solving a convex optimization problem

$$DI(\gamma) \triangleq \min_{\tilde{\Gamma}} \limsup_{T\to+\infty} \frac{1}{T} I(\mathbf{x}_0^{T-1} \to \mathbf{u}_0^{T-1}) \quad (5a)$$

$$\text{s.t.} \limsup_{T\to+\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}c(\mathbf{x}_t, \mathbf{u}_t) \leq \gamma \quad (5b)$$

where $I(\mathbf{x}_0^{T-1} \to \mathbf{u}_0^{T-1}) \triangleq \sum_{t=0}^{T-1} I(\mathbf{x}_0^t; \mathbf{u}_t|\mathbf{u}_0^{t-1})$ is called *directed information*. Here, we employ a definition of directed information by Massey (1990), while several generalized definitions are proposed in the literature. Minimization in (5a) is over the space $\tilde{\Gamma}$ of the sequences of stochastic kernels $\{p_t(u_t|x_0^t, u_0^{t-1})\}_{t=0,1,\ldots}$.

Silva et al. (2011) also show that $R(\gamma)$ is upper bounded by $DI(\gamma)+c$, where the c is a scalar constant. Their proof is based on the construction of an entropy-coded dithered quantizer (ECDQ) whose rate loss due to the space-filling loss of the lattice quantizers and the loss of entropy coders is bounded by c. This result, although restricted to SISO control systems in the original paper, is recently extended to MIMO control systems by Tanaka et al. (2016). For completeness, we present the synthesis procedure proposed there in Section 3.

Although the quantizer/controller design methods proposed in Silva et al. (2011) and Tanaka et al. (2016) are suboptimal (with bounded performance loss), they are relatively simple to implement, since they are built on uniform quantizers. The optimal quantizer/controller design is much more involved; iterative design algorithms and structural results are studied by Bao et al. (2011) and Yüksel (2014).

The purpose of this paper is to demonstrate that the performance of the design in Silva et al. (2011) and Tanaka et al. (2016) can be further improved by choosing the *block-length* [3] optimally. If the block-length is k, the state information is quantized, encoded, and transmitted to the controller only once in every k time steps. In other time steps, no information is transmitted from the encoder to the controller, i.e., $\boldsymbol{\ell}_t = 0$ for t such that $t \bmod k \neq 0$. This is visualized in Figure 2.

There are pros and cons of selecting block-lengths larger than one. They can be intuitively seen in the following trade-off:

(A) From a control-theoretic viewpoint, selecting large block-lengths is a disadvantage. Notice that control input $\mathbf{u}_t$ can depend on the state sequence only up to $\mathbf{x}_{t'}$, where t' is the largest integer such that $t' \leq t$ and $t' \bmod k = 0$. In particular, large block-length implies large delay.

(B) From an information-theoretic viewpoint, transmitting high-resolution data once in a while rather than transmitting low-resolution data at every time step is beneficial, since the former allows more efficient data compression. Namely, the effect of space-filling loss due to quantization and the entropy coding loss per time step can be made smaller.

In this paper, we perform a quantitative study on this trade-off using theoretical lower and upper bounds of the best achievable rates. Since the best achievable rates are still difficult to evaluate exactly, it remains difficult to determine the optimal block-length analytically. Hence, we also perform a numerical study to see how the performance of Tanaka et al. (2016) varies with different block-lengths.

2. RELATED WORK

LQG control problems with data-rate constraints have been tackled by many papers from various angles. Joint controller and quantizer design was considered by Bao et al. (2011) and Yüksel (2014). Extensions of the classical separation principle are discussed in Matveev and Savkin (2004), Fu (2009) and You and Xie (2011). Rate-performance trade-off studies can be found in Tatikonda et al. (2004), Huang et al. (2005), Lemmon and Sun (2006) and Freudenberg et al. (2011). In particular, Silva et al. (2011) establishes a connection between (4) and (5). Many important results that cannot be mentioned here can be found in Yüksel and Başar (2013).

Optimal block-length for data-rate minimization in LQG control is considered in Borkar and Mitter (1997). A similar problem for controlled hidden Markov chain is considered in Tan et al. (2004). Although these papers are closest in spirit to this paper, these analyses involve computationally intractable steps (e.g., dynamic programming) for the block-length optimization, which restrict venues for quantitative studies. In this paper, we apply newly obtained theoretical upper and lower bounds (Silva et al. (2011)) and synthesis (Tanaka et al. (2016)) to make the study more quantitative and computationally accessible.

In this paper, we assume time-triggered communications between encoder and controller. However, average data-rate minimization may also be attained by event-triggered communications. Åström and Bernhardsson (2002) compared conventional periodic (Riemann) sampling and event-based (Lebesgue) sampling strategies, and showed that the latter achieves a better performance in a certain context. This observation is extended by Cervin and Henningsson (2008) to the situations in which multiple control loops share a communication medium. Information-theoretic framework for event-triggered control is recently considered by Tallapragada and Cortés (2014).

3. NETWORKED CONTROLLER DESIGN

In this section, we summarize the result obtained by Tanaka et al. (2016). A control architecture proposed there for problem (4) (Figure 1) is shown in Figure 3. Components in Figure 3 can be constructed by the following procedure.

[3] In this paper *block-length* is used as a synonym for *data transmission intervals* or *sampling intervals*.