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Complexities and Performance Limitations in Growing Time-Delay Noisy Linear Consensus Networks

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Abstract: This work investigates topology design for optimal performance in time-delay noisy networks. Performance of the network is measured by the square of the system's $\mathcal{H}_2$-norm. The focus of this paper is on adding new interconnections to enhance performance of the time-delay first-order connected consensus network. We discuss complexity of topology optimization for delayed networks and develop two practical methods to tackle the combinatorial eigenvalue problem without exhaustive search or eigen-decomposition. Furthermore, we compare these methods and discuss their degrees of optimality.

Keywords: Consensus Networks, Topology Design, Time-Delay Systems, Network Performance Measures

1. INTRODUCTION

Our objective is to enhance $\mathcal{H}_2$-norm performance of a noisy time-delay linear consensus network by establishing new feedback interconnections.

Literature Review: In recent years, there has been an immense interest in studying susceptibility of networked systems to input disturbances. Bamieh et al. (2012) quantified the $\mathcal{H}_2$-norm of first-order consensus networks as a measure of robustness with respect to stochastic disturbances. With growing applications of consensus protocols in various disciplines, network topology design for such systems turned into a class of challenging and demanding problems. These applications range from coordinating networks of autonomous vehicles and sensor networks to synchronous oscillators in power networks [Olfati-Saber et al. (2007)]. Over the years, the concept of improving $\mathcal{H}_2$-norm of first-order consensus networks is manifested through different manuscripts. Ghosh et al. (2008) studied optimal weight allocation in graphs to minimize total effective resistance. With $\mathcal{H}_2$-norm square of first-order consensus networks being proportional to total effective resistance of its underlying graph, Siami et al. (2016), Summers et al. (2015), and Moghaddam and Jovanovic (2015) study the problem of establishing new interconnections under different constraints and scenarios. Olfati-Saber and Murray (2004) introduced algebraic connectivity of the network's underlying graph as a measure of performance for first-order consensus networks. Mosk-Aoyama (2008) proved the problem of adding a prespecified number of edges to a network, maximizing its algebraic connectivity, is Np-hard. However, Ghosh and Boyd (2006) suggested an efficient heuristic using the Fiedler vector of the graph to address the combinatorial problem. Other measures of performance for first-order consensus networks are discussed in Siami and Motee (2014), Siami and Motee (2016).

Despite extensive research on consensus network performance in absence of delay, there have been limited attention to such measures for consensus networks in presence of delay. Ghaedsharaf et al. (2016) quantified $\mathcal{H}_2$-norm performance and some fundamental limits on the best achievable performance of time-delay first-order consensus networks in terms of its underlying graph spectrum. Nevertheless, most of the efforts in the literature study conditions for stability of time-delay systems or convergence time rather than quality of consensus in terms of dissipated energy to reach consensus [Olfati-Saber and Murray (2004), Münz et al. (2010), Nedić and Ozdaglar (2010), Somarakis and Baras (2015)].

Rafiee and Bayen (2010) aim to design the optimal topology for consensus networks in presence of time-delay considering algebraic connectivity of the underlying graph as a performance measure. However, for time-delay consensus networks, the second smallest eigenvalue of the Laplacian is not a good measure of performance [Ghaedsharaf et al. (2016)]. Qiao and Sipahi (2014) consider designing delay dependent coupling weights in order to ensure stability in the presence of a integer-valued homogeneous time-delay.

A major issue in time-delay consensus networks, which emerges in the problem of adding new interconnections, is that new interconnections might deteriorate the performance or even stability of the network. On the contrary, in the absence of delay, new links will improve the $\mathcal{H}_2$ performance measure because of the monotonicity property [Siami et al. (2016)]. Thus, designing topology for time-delay networks is a more delicate task compared to the situation that time-delay is absent.

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Our Contribution: In light of Ghaedsharaf et al. (2016), in Section 3 we formulate the problem of adding new interconnections to the time-delay first-order consensus network. Further, in Section 4, we find a highly accurate approximation function that would spare us from the complexity of eigen-decomposition of the Laplacian matrix. Moreover, by utilizing that approximate function, we find the semidefinite programming (SDP) relaxation of the problem. In addition, we find bounds on the optimality degree if we use the approximate function instead of the original function. Notwithstanding that the SDP is a relaxation of the main problem, it provides us with a good lower bound on the best achievable performance. In Section 5 we take advantage of the approximate function that we introduced in Section 4 and we propose a greedy algorithm to tackle our combinatorial problem. Furthermore, we show that our approximate cost function has the submodular property which allows us to find bounds on the optimality of our greedy method. Lastly, Section 6 is devoted to numerical examples for demonstrating the utility of our results. All the proofs of this paper are omitted due to space limitations.

2. PRELIMINARIES AND DEFINITIONS

2.1 Basic Definitions

Throughout the paper the following notations will be used. We denote the transpose and conjugate transpose of matrix A by A^{T} and A^{H}, respectively. Also, the set of non-negative (positive) real numbers is indicated by $\mathbb{R}_+(\mathbb{R}_{++})$. An undirected weighted graph $\mathcal{G}$ is denoted by the triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, w)$ where $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ is the set of nodes of the graph, $\mathcal{E}$ is the set of links of the graph and $w : \mathcal{E} \to \mathbb{R}_{++}$ is the weight function that maps each link to a positive scalar. We let L to be the Laplacian of the graph, defined by $L = \Delta - A_{\mathcal{G}}$, where Δ is the diagonal matrix of node degrees and $A_{\mathcal{G}}$ is the adjacency matrix of the graph. Each node i at time t has a scalar state which is $x_i(t)$. Vector of all ones in $\mathbb{R}^n$ is denoted by $\mathbf{1}_n$ and $J_n = \mathbf{1}_n \mathbf{1}_n^{\mathrm{T}}$. Furthermore, we let $M_n = I_n - \frac{1}{n} J_n$ and we refer to it as the centering matrix. For an undirected graph with n nodes, the resulting Laplacian eigenvalues are real and showed by $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Also, for a given Laplacian matrix L, we denote by $L^{\dagger} = [l_{ij}^{\dagger}]$ the Moore-Penrose pseudo-inverse of L and by $r(L) = [r_{ij}(L)]$ the effective resistance matrix, with entries $r_{ij}(L) = l_{ii}^{\dagger} + l_{jj}^{\dagger} - l_{ij}^{\dagger} - l_{ji}^{\dagger}$. Additionally, we use $\mathbf{diag}()$ as the operator that maps a vector $y \in \mathbb{R}^n$ to a diagonal square matrix $Y \in \mathbb{R}^{n \times n}$ with components of y on the main diagonal of Y. In addition, we denote maximum degree of nodes of the graph, by $d_{\max}$. Also, for a given function f we denote its supremum in its domain by $\|f\|_{\infty}$.

Definition 1. $g : \mathbb{R}^n \to \mathbb{R}$ is a Schur-convex function if for every doubly stochastic matrix $D \in \mathbb{R}^{n \times n}$ and all $x \in \mathbb{R}^n$,

$$g(Dx) \leq g(x). \tag{1}$$

Definition 2. Let S be a finite set. A function $h\colon 2^S \to \mathbb{R}$ is sub-modular if for any $S_1 \subset S_2 \subset S$ and $x \in S \setminus S_2$,

$$h(S_1 \cup \{x\}) - h(S_1) \geq h(S_2 \cup \{x\}) - h(S_2). \tag{2}$$

2.2 Time-delay noisy linear consensus networks

In this paper, we consider a class of linear consensus networks in which each node resembles a subsystem with a scalar state variable. We assume that each node has a delay in reacting to other nodes or has a delay in computing or accessing its own state. Thus, the states of each subsystem in absence of external disturbances evolves through the following differential equation

$$\dot{x}_i(t) = \sum_{j \neq i} a_{ij} \big(x_j(t - \tau) - x_i(t - \tau)\big), \tag{3}$$

where $\tau \geq 0$ is the delay parameter and a_{ij} is the ij^{th} component of the adjacency matrix from the underlying coupling graph. Thus, we assume that the delay affects both the neighbors and the node itself. Consequently, we study the following first-order consensus network with n nodes having homogeneous delay and underlying graph Laplacian L:

$$\mathcal{N}(L; \tau) : \begin{cases} \dot{x}(t) = & -Lx(t - \tau) + \xi(t), \\ y(t) = & M_n x(t), \end{cases} \tag{4}$$

where $\xi(t) \in \mathbb{R}^n$ is a vector of independent Gaussian white noise process with zero mean and identity covariance, i.e.,

$$\mathbb{E}\left[\xi(t_1)\xi^{\mathrm{T}}(t_2)\right] = I_n \delta(t_1 - t_2), \tag{5}$$

and $\delta(t)$ is the Dirac's delta function.

Definition 3. The $\mathcal{H}_2$-norm squared performance of the dynamical system (4) from input ξ to output y is defined as the following quantity

$$\begin{aligned} \rho_{\mathrm{ss}}(L; \tau) &= \lim_{t \to \infty} \mathbb{E}\left[\sum_{i=1}^{n} (x_i(t) - \bar{x}(t))^2\right] \\ &= \lim_{t \to \infty} \mathbb{E}\left[y^{\mathrm{T}}(t) y(t)\right] \end{aligned} \tag{6}$$

where $\bar{x}(t)$ is the following scalar

$$\bar{x}(t) = \frac{1}{n} \sum_{i=1}^{n} x_i(t). \tag{7}$$

Theorem 4. (Ghaedsharaf et al. (2016)). In presence of a time-delay $\tau \geq 0$, the performance measure of the linear consensus network (4), can be written as an additively separable function of eigenvalues from the Laplacian matrix. In other words, we have the following

$$\rho_{\mathrm{ss}}(L; \tau) \;=\; \sum_{i=2}^{n} f_{\tau}(\lambda_i), \tag{8}$$

where

$$f_{\tau}(\lambda_i) \;:=\; \frac{1}{2\lambda_i} \frac{\cos(\lambda_i \tau)}{1 - \sin(\lambda_i \tau)} \tag{9}$$

and λ_i for $i = 2, \ldots, n$ are Laplacian eigenvalues of the underlying graph.

Corollary 5. For a fixed underlying graph, performance of the consensus network (4) is an increasing function of time-delay, i.e., for $0 \leq \tau_1 < \tau_2 < \frac{\pi}{2\lambda_n}$, we have the following