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Distributed Feedback Control of Quantum Networks ⋆

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Abstract: In this paper, we deal with a distributed feedback control of quantum networks called a quantum consensus algorithm (QCA) with local quantum observation and feedback proposed by Kamon & Ohki (2013, 2014) and prove strictly that QCA makes quantum states converge to a quantum state called symmetric state consensus (SSC) with probability one from arbitrary initial states keeping purity. The difficulty of the proof is from that the objective system is stochastic and non-linear, and we solve it by employing the stochastic Lyapunov stability analysis. We also show that QCA can generate a desirable W-state, which is known as an important entangled quantum state and utilized for many applications of quantum information technology.

Keywords: quantum feedback, quantum networked system, stochastic system, quantum consensus, distributed feedback

1. INTRODUCTION

Quantum control has been actively investigated to overcome such problems as the generation or preservation of quantum bits (qubits) under noisy environments Wiseman & Milburn (2009). From the establishment of quantum filtering theory Belavkin (1992), research about quantum control has advanced and contributed to broad areas of quantum information technologies Mirrahimi & van Handel (2007). However, as is the case with classical systems, it is quite difficult to control quantum bits when the number of bits is large because of the increasing complexity of instrument networks (e.g., see Yokoyama et al. (2013) for the case of optical systems). Then, a distributed operation called quantum consensus, which is one of the distributed quantum information applications, is a promising idea to generate quantum states of large-scale quantum systems.

Mazzarella et al. (2013) have developed a framework of quantum consensus as the extension of classical consensus problems. They have defined several types of quantum consensus states, derived their hierarchical relationship and proposed a quantum version of gossip algorithm which asymptotically generates a consensus state called symmetric state consensus (SSC).

Their algorithm can be regarded as an autonomous system like classical consensus systems and it contains no feedback input operation depending on the current quantum states. Then, in fact, Kamon & Ohki (2013) proved that the algorithm loses the purity of quantum states during the consensus operations. Purity is an important quantity for application to quantum information technology and above fact is not desirable for the purpose of generating useful quantum states. The similar approaches for consensus with no feedback control (e.g. Sepulchre et al. (2010); Shi et al. (2015, 2016)) also have this issue.

Motivated by the above fact, Kamon & Ohki (2013) and Kamon & Ohki (2014) have proposed a hybrid type of the distributed consensus algorithm and a distributed feedback with quantum state observation; projective measurements, in order to realize SSC and high purity simultaneously. They have shown the efficiency of their control scheme by numerical simulations and further expected that their algorithm realizes artificial bosonization or artificial fermionization.

The convergence of their algorithm, however, has not been proved and left for as an open problem. Then, in this paper, we tackle with this open problem and solve it completely. For more details, we modify the algorithm proposed by Kamon & Ohki (2013, 2014) and give a strict proof of the convergence to SSC from arbitrary initial states keeping purity. The difficulty of the proof is from that the dynamics is governed by two types of stochastic processes; (1) probabilistic selection of local subsystems among the whole networked quantum system, (2) feedback control action depending on the probabilistic quantum observation results, projective measurements, of the selected local subsystems. Therefore, the feedback control dynamics depends on the state-depending complicated combinations of the above stochastic processes and, as a result, it is represented as "a stochastic non-linear equation." In fact, a simple idea of applying the Kraus map discussed in Mazzarella et al. (2013) for their "deterministic linear autonomous systems" is not applicable in our case and its analysis requires strict dealing with the dynamics and the combinations as discussed in our paper.

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To overcome the complexity of stochastic non-linear systems, we employ the stochastic version of the Lyapunov stability theory, which is a well known method in quantum control Mirrahimi & van Handel (2007). Moreover, we also show that the proposed algorithm can generate a W-state, which cannot be obtained by the algorithm of Mazzarella *et al.* because the purity of the W-state is maximum among all the quantum states. It is well known that the W-state is one of the significant quantum entangled states and utilized in wide areas such as quantum memory. This is an important application of quantum consensus generation.

The similar research on the quantum consensus with feedback control is in Mazzarella et al. (2015), where the target state is restricted to an eigenstate and entangled states are not realized. Ticozzi (2016) recently reports the similar result of this paper in a deterministic way, however it is unclear whether its assumed deterministic operation can be realized by a feedback strategy which essentially depends on the probabilistic observation output by projective measurements.

This paper is organized as follows. In Section 2, we introduce some mathematical preliminaries and define the problem setting. In Section 3, we show the main results of this paper and prove them. In Section 4, we show numerical examples to confirm the efficiency of the proposed algorithm. Finally, we conclude this paper in Section 5.

Note that we omit many of the proofs for lemmas in this paper from the page limitation.

2. FORMULATION

2.1 Convergence of Stochastic System

Let $\{x_n\}_{n\in\{0\}\cup\mathbb{N}} \subset \mathbb{C}^m$ be a sequence of random variables. Then, we introduce definitions of convergence as follows:

Definition 1. A sequence $\{x_n\}$ is said to converge to $\tilde{x}$ in probability if $\lim_{n\to\infty} \mathbb{P}\{|x_n - \tilde{x}| \geq \epsilon\} = 0$ for any $\epsilon > 0$.

Definition 2. A sequence $\{x_n\}$ is said to converge to $\tilde{x}$ with probability one (w.p.1) if $\mathbb{P}\{\lim_{t\to\infty} x_n = \tilde{x}\} = 1$.

It is known that convergence w.p.1 is stronger than convergence in probability.

In this paper, we deal with a quantum control system which makes a quantum state converge to a target state in the above probabilistic sense. In order to show such convergence, we employ the following stochastic Lyapunov stability theorem.

Definition 3. A set $\mathcal{C}$ is called an invariant set if any initial state of a dynamical system belonging to $\mathcal{C}$ never leaves $\mathcal{C}$.

Proposition 1. (Kushner (1971)) Let $\{x_n\}_{n\in\{0\}\cup\mathbb{N}} \subset \mathbb{C}^m$ be a state of some dynamical system and a Markov process. Assume that there exist bounded non-negative functions $V(x)$ and $k(x)$ which satisfy

$$\mathbb{E}\{V(x_n)|x_{n-1}\} - V(x_{n-1}) = -k(x_{n-1}) \tag{1}$$

for all $n \in \mathbb{N}$. Then, $k(x_n) \to 0$ $(n \to \infty)$ for almost all the paths. In addition, let $\mathcal{M} = \{x \in \mathbb{C}^m \mid k(x) = 0\}$, and let $\tilde{\mathcal{M}}$ be the largest invariant set of $\mathcal{M}$, then x_n converges to $\tilde{\mathcal{M}}$ in probability.

In some cases, convergence in probability implies convergence w.p.1.

2.2 Quantum State and Quantum Consensus State

In this paper, we deal with a multipartite quantum system composed of N isomorphic subsystems, labeled with indices $i = 1, 2, \ldots, N$, with associated Hilbert space $\mathcal{H}^N := \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_N$, with $\dim(\mathcal{H}_i) = D$ for all i and D is an integer satisfying $D \geq 2$. Let $\{|d_i\rangle\}_{d_i\in\{0,1,\ldots,D-1\}}$ be a set of basis vectors of $\mathcal{H}_i$, then the basis vectors of $\mathcal{H}^N$ are represented by $\{|d_1\rangle\otimes|d_2\rangle\otimes\cdots\otimes|d_N\rangle\}_{\forall i, d_i\in\{0,1,\ldots,D-1\}}$. Hereafter, we abbreviate $|d_1\rangle\otimes|d_2\rangle\otimes\cdots\otimes|d_N\rangle$ to $|d_1 d_2 \cdots d_N\rangle$. In addition, we regard $\mathcal{H}_i$ as $\mathbb{C}^D$ and the basis vector $|d_i\rangle$ as $(0\ 0\ \cdots\ 0\ 1\ 0\ 0\ \cdots\ 0)^\top$, i.e., the d_i+1-th element is one and the others are zero, where $\top$ is a transpose operator.

Remark 1. Our main results are obtained in the case of $D = 2$, while some of lemmas are also true in the general case. So, we specify the condition $D = 2$ only if necessary in the following.

Define $\mathcal{B}(n)$ as a set of matrices with dimension $n \times n$. Then a quantum state on $\mathcal{H}^N$ is represented by a density matrix in

$$\mathfrak{D}(D^N) := \{\rho \in \mathcal{B}(D^N)|\ \rho = \rho^\dagger \succeq 0, \mathrm{tr}(\rho) = 1\}, \tag{2}$$

where $\dagger$ is the complex conjugate transpose operator, $\succeq 0$ means that the matrix is semi-positive definite, and $\mathrm{tr}(\cdot)$ is an trace operator. A density matrix completely represents a probability distribution of a quantum system.

In particular, if the rank of a density matrix is one, the quantum state is called a pure state, while it is called a mixed state if the rank is larger than one. A pure state ρ is completely expressed by a state vector ψ as $\rho = \psi\psi^\dagger$, which is an element of $\mathfrak{D}'(D^N) := \{\psi \in \mathbb{C}^{D^N} \mid \|\psi\| = 1\}$, where $\|\cdot\|$ is 2-norm.

In this paper, we deal with a networked quantum system and consider to realize a quantum consensus state, called symmetric state consensus (SSC) introduced by Mazzarella et al. (2015) as follows:

Definition 4. (Mazzarella et al. (2015)) Let π be a permutation of integers $1, 2, \ldots, N$, and let $U_\pi \in \mathcal{B}(D^N)$ be a permutation matrix satisfying $U_\pi(x_1 \otimes x_2 \otimes \cdots \otimes x_N) = x_{\pi(1)} \otimes x_{\pi(2)} \otimes \cdots \otimes x_{\pi(N)}$ for all $\{x_n\}_{n=1}^N \subset \mathbb{C}^D$. Then, a quantum state $\rho \in \mathfrak{D}(D^N)$ is called in symmetric state consensus (SSC) if $U_\pi \rho U_\pi^\dagger = \rho$ holds with any π.

We also use the notation SSC to represent the set of all the quantum states in symmetric state consensus.

2.3 Network Structure and Quasi-local Operation

Mazzarella et al. (2015) introduced a consensus algorithm to realize SSC, however Kamon & Ohki (2013) proved that it loses purity

$$\mathrm{tr}(\rho^2). \tag{3}$$

Motivated by this fact, in the following of this section, we introduce a networked quantum system composed of several quantum subsystems with a hybrid type (Kamon & Ohki (2013, 2014)) of a quantum consensus algorithm (Mazzarella et al. (2015)) and feedback control with observations, that is, distributed quasi-local measurements and