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Feedback Control of a Thermal Fluid Based on a Reduced Order Observer ⋆

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Abstract: We discuss the problem of designing a feedback control law based on a reduced order observer, which locally stabilizes a two dimensional thermal fluid modeled by the Boussinesq approximation. We consider mixed boundary control for the Boussinesq equations in an open bounded and connected domain. In particular, the controllers are finite dimensional and act on a portion of the boundary through Neumann/Robin boundary conditions. A linear Luenberger observer is constructed based on point observations of the linearized Boussinesq equations. The current setting of the system leads to a problem with unbounded control inputs and outputs. Linear Quadratic Gaussian (LQG) balanced truncation is employed to obtain the reduced order model for the linearized system. The feedback law can be obtained by solving an extended Kalman filter problem. The numerical results show that the nonlinear system coupled with the reduced order observer through the feedback law is locally exponentially stable.

1. INTRODUCTION

We consider the problem of feedback stabilization of a two dimensional thermal fluid. The transport of thermal energy in a viscous incompressible fluid can be modeled by the Boussinesq approximation, which couples the Navier-Stokes equations with the convection-diffusion equation for the temprature of the fluid. Feedback control for fluid flows is a very active area and has been widely studied; see, e.g., Choi et al. (1993); Burns et al. (1998); Wang (2003); Barbu et al. (2006); Lee and Choi (2006); Raymond (2006, 2007); Raymond and Thevenet (2010); Badra (2012); Bänsch and Benner (2012); Nguyen and Raymond (2015); Brunton and Noack (2015). Our current work focuses on the low order stabilizing feedback control design that is implementable in real time.

Recent work in Burns et al. (2016) considered the LQR control design for a two dimensional Boussinesq equations. It is considered that the control inputs are finite dimensional and act on a portion of the boundary through Neumann/Robin boundary conditions. Dirichlet boundary conditions are imposed on the rest of the boundary. The standard Riccati-based feedback law can be derived. Numerical experiments show that the Riccati-based feedback law locally exponentially stabilizes an unstable steady state solution to the nonlinear Boussinesq system. However, full state feedback control is not practical for most flow control applications.

In this work, we continue to use the setup in Burns et al. (2016), but consider the problem of stabilizing a possible unstable steady state solution to the nonlinear Boussinesq equations by a feedback control law based on a reduced order observer. In particular, point observations are used for the output measurement and a linear Luenberger observer design is employed for the the state estimation of the linearized Boussinesq system. The current setting naturally leads to a problem with unbounded control inputs and outputs. To obtain a reduced order observer, we need an effective reduced order model for the unbounded input-output system.

For model reduction of linear systems, a well-known class of methods with excellent properties are the balanced truncation algorithms (Antoulas, 2005; Zhou et al., 1996). The fundamental algorithm in this class, (standard) balanced truncation, is only applicable to exponentially stable systems. For unstable systems with no eigenvalues on the imaginary axis, a generalization of balanced truncation was introduced in Zhou et al. (1999). Ideas from this work have recently been used for generalized balanced truncation model reduction of unstable systems derived from a spatial discretization of a linear PDE system; see, e.g., Ahuja and Rowley (2010); Benner et al. (2016); Flinois et al. (2015) and the references therein.

Another balanced truncation approach that is directly applicable to unstable systems is LQG balanced truncation. This method is derived from the algebraic Riccati equations arising in LQG feedback control, and therefore LQG balanced truncation has been frequently used for model reduction in feedback control applications. This method has been applied to compute reduced order controllers for many PDE systems; see, e.g., Batten and Evans (2010); Benner and Heiland (2015); Breiten and Kunisch (2015, 2016); Evans (2003); Singler and Batten (2009). We use

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this approach here to develop a linear reduced order controller for the nonlinear Boussinesq system.

We note that a reduced order stabilizing feedback controller for the Navier-Stokes equations was designed using LQG balanced truncation in Benner and Heiland (2015); however, boundary control was not considered in that work. The main contribution of this work is the investigation of the performance of a low order LQG balanced feedback controller for the Boussinesq equations with boundary control.

2. THE MODEL

Let Ω be an open bounded and connected domain with a Lipchitz boundary Γ. The Boussinesq approximation is given by

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = \frac{1}{Re}\text{div }(\nabla\mathbf{v}+(\nabla\mathbf{v})^T) - \nabla p + \bar{\mathbf{e}}\frac{Gr}{Re^2}\theta + f_{\mathbf{v}}, \tag{1}$$

$$\text{div } \mathbf{v} = 0, \tag{2}$$

$$\partial_t\theta + \mathbf{v}\cdot\nabla\theta = \frac{1}{RePr}\Delta\theta + f_\theta, \tag{3}$$

where $\mathbf{v}(x,t)$ is the velocity, $p(x,t)$ is the pressure, $\theta(x,t)$ is the fluid temperature, Re is the Reynolds number, Gr is the Grashof number, Pr is the Prandtl number, and $\bar{\mathbf{e}} = [0,1]^{\mathrm{T}}$ is the gravitational force direction. We assume $f_{\mathbf{v}}$ is a time independent external body force and f_θ is a time independent heat source density. Consider a 2D domain shown in Figure 1. Assume that the controlled

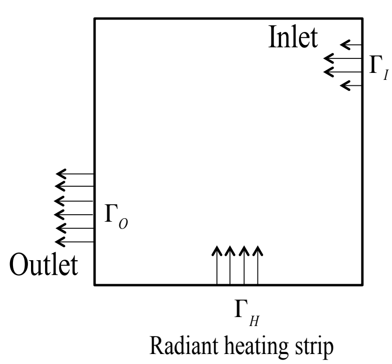

Fig. 1. 2D domain

airflow is coming in through the inlet Γ_{I}, which is a subset of the boundary Γ, with Robin boundary control for both velocity and temperature. The airflow exits at the outlet Γ_{O} with stress-free fluid and natural (or unforced) convective flux boundary conditions. In addition, there is a radiant heating strip, denoted by Γ_{H}, on the floor with Neumann boundary control for temperature. We impose no slip boundary conditions for the velocity on $\Gamma_{\mathrm{f}} = \Gamma\setminus(\Gamma_{\mathrm{I}}\cup\Gamma_{\mathrm{O}})$ and zero Dirichlet boundary condition for temperature on $\Gamma_{\mathrm{D}} = \Gamma\setminus(\Gamma_{\mathrm{I}}\cup\Gamma_{\mathrm{O}}\cup\Gamma_{\mathrm{H}})$. Here the boundaries Γ_{I}, Γ_{O} and Γ_{H} are disjoint. The boundary conditions can be formulated as follows.

$$(\mathcal{T}(\mathbf{v},q)\cdot n + \alpha\mathbf{v})|_{\Gamma_{\mathrm{I}}} = \sum_{i=1}^m (b_{\mathbf{v}_i}|_{\Gamma_{\mathrm{I}}})(x)u_{\mathbf{v}_i}(t), \tag{4}$$

$$\mathcal{T}(\mathbf{v},q)\cdot n|_{\Gamma_{\mathrm{O}}} = 0, \quad \mathbf{v}|_{\Gamma_f} = 0, \tag{5}$$

$$(\frac{1}{RePr}\frac{\partial\theta}{\partial n} + \beta\theta)|_{\Gamma_{\mathrm{I}}} = \sum_{i=1}^m (b_{\theta_{\mathrm{I}_i}}|_{\Gamma_{\mathrm{I}}})(x)u_{\theta_{\mathrm{I}_i}}(t), \tag{6}$$

$$\frac{1}{RePr}\frac{\partial\theta}{\partial n}|_{\Gamma_{\mathrm{O}}} = 0, \tag{7}$$

$$\frac{1}{RePr}\frac{\partial\theta}{\partial n}|_{\Gamma_{\mathrm{H}}} = \sum_{i=1}^m (b_{\theta_{\mathrm{H}_i}}|_{\Gamma_{\mathrm{H}}})(x)u_{\theta_{\mathrm{H}_i}}(t), \quad \theta|_{\Gamma_{\mathrm{D}}} = 0, \tag{8}$$

where $\mathcal{T}(\mathbf{v},p)$ is the fluid Cauchy stress tensor defined by

$$\mathcal{T}(\mathbf{v},q) = \frac{1}{Re}(\nabla\mathbf{v}+(\nabla\mathbf{v})^T) - qI.$$

For $u_{\mathbf{v}_i} = u_{\theta_j} = 0$, let $(\mathbf{v}_e, p_e, \theta_e)$ be a steady-state (equilibrium) solution to equations (1)–(3). Notice that for large Reynolds numbers or strong external body forces, the steady-state solution can be unstable. We introduce the new variables

$$\mathbf{w} = \mathbf{v} - \mathbf{v}_e, \quad T = \theta - \theta_e \quad \text{and} \quad q = p - p_e.$$

Then the translated system is given by

$$\frac{\partial\mathbf{w}}{\partial t} = \frac{1}{Re}\Delta\mathbf{w} - \mathbf{w}\cdot\nabla\mathbf{v}_e - \mathbf{v}_e\cdot\nabla\mathbf{w} - \mathbf{w}\cdot\nabla\mathbf{w} - \nabla q + \bar{\mathbf{e}}\frac{Gr}{Re^2}T, \tag{9}$$

$$\nabla\cdot\mathbf{w} = 0, \tag{10}$$

$$\frac{\partial T}{\partial t} = \frac{1}{RePr}\Delta T - \mathbf{w}\cdot\nabla\theta_e - \mathbf{v}_e\cdot\nabla T - \mathbf{w}\cdot\nabla T. \tag{11}$$

Let $\mathbf{x}(t) = [\mathbf{w}(t), T(t)]^T$. Then the controlled translated equations (1)–(7) can be rewritten as

$$\dot{\mathbf{x}}(t) = \mathbf{A}_e\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{F}(\mathbf{x}), \tag{12}$$

where $\mathbf{A}_e$ is the translated linearized system operator, $\mathbf{B}$ is the boundary control operator and $\mathbf{F}$ is the nonlinear mapping. The details for the formulation of (12) can be found in Burns et al. (2016). By the linearized theory of hydrodynamic stability in Sattinger (1973), the stability of $(\mathbf{v}_e, p_e, \theta_e)$ is determined by the spectrum of $\mathbf{A}_e$ associated with the boundary conditions.

Linearizing system (12) yields

$$\dot{\mathbf{z}}(t) = \mathbf{A}_e\mathbf{z}(t) + \mathbf{B}\mathbf{u}(t), \tag{13}$$

in a Hilbert space $\mathcal{H}$, where $\mathbf{z}(t) = [\mathbf{w}(t), T(t)]^T$. The control space is $\mathcal{U} = \mathcal{R}^{3m}$. Moreover, consider the output measurement of the linearized system (13) by point observations

$$\mathbf{y}(t) = \mathbf{C}\mathbf{z}(t) = [\mathbf{z}(\xi_1,t), \mathbf{z}(\xi_2,t), \ldots, \mathbf{z}(\xi_n,t)]^T \in \mathcal{R}^{3n}, \tag{14}$$

where $\mathbf{C} = [\delta(x-\xi_1), \delta(x-\xi_2), \ldots, \delta(x-\xi_n)]^T$ is the point observation operator and $\xi_i \in \Omega, i = 1, 2\ldots, n$, are the points of observation. Next we apply the linear Luenberger observer design to system (13)–(14). Consider

$$\dot{\mathbf{z}}_c = \mathbf{A}_e\mathbf{z}_c + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{C}\mathbf{z}_c - \mathbf{y}), \tag{15}$$

where $\mathbf{L}$ is called the filtering operator. Note that $\mathbf{A}_e$ generates an analytic C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{H}$. The resolvent set of $\mathbf{A}_e$ contains a sector. Thus, there are at most a finite number of eigenvalues of $\mathbf{A}_e$ in the right complex half-plane $\{\lambda \in \mathbf{C} : \text{Re}\lambda \geq 0\}$. Therefore, there exists a real number $\lambda_0 \in \rho(\mathbf{A}_e)$, sufficiently large, such that $\lambda_0\mathbf{I} - \mathbf{A}_e$ is a strictly positive operator and the fractional powers $(\lambda_0\mathbf{I} - \mathbf{A}_e)^\sigma$ are well defined for $0 \leq \sigma \leq 1$. In addition, since $\mathbf{B}$ is a Neumann/Robin type boundary