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Optimal Control of the Viscous Burgers Equation by the Hopf-Cole Transformation and Its Properties

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Abstract: This paper discusses an optimal control problem for the viscous Burgers equation using the Hopf-Cole transformation. Using this transformation, an optimal control problem for the Burgers equation is transformed into that for a linear heat equation. Although this transformation makes the boundary condition complicated, we use a state feedback to overcome the difficulty. Using the state feedback, an exact solution of the heat equation is obtained by Laplace transformation which enables us to obtain the optimal control input for the heat equation exactly. In fact, it is obtained by a solution of a well-known Fredholm integral equation. Furthermore, we discuss how to choose an achievable desired terminal state from the controllability point of view. In addition, we exhibit the effectiveness of the proposed approach through a numerical simulation. It verifies that when the solution of the heat equation at terminal time converges to its desired value, that of the corresponding viscous Burgers equation also converges to the desired one.

Keywords: Distributed parameter systems, optimal control, state feedback.

1. INTRODUCTION

Partial differential equations (PDEs) have an important role in physics since various phenomena are described by them. Control schemes for those distributed parameter systems have been developed. Among PDEs, we focus on the viscous Burgers equation, which is considered as a one dimensional model of incompressible Navier-Stokes equations. In addition, the viscous Burgers equation has a steady shock-like solution, so this equation can be used as a model for a one dimensional traffic flow model (Nagase and Sakawa (1979)), or a Jet noise model. Furthermore, this equation is used for benchmark problems of the nonlinear control for distributed parameter systems. The viscous Burgers equation can be solved exactly by the Hopf-Cole Transformation (Hopf (1950)). Using it, the viscous Burgers equation is transformed to a linear heat equation. Global controllability, local controllability, and approximate controllability are discussed for the viscous Burgers equation (Krstic (1999), Guerrero and Imanuvilov (2007), Diaz (1995)).

Control methods for the Burgers equation have been proposed by several different researchers using different approaches. One is to approximate an infinite dimensional system by a finite dimensional one. In (Hamaguchi et al. (2015)), the combination of the Proper Orthogonal Decomposition (POD)-Galerkin method and the stable manifold method are applied to nonlinear distributed parameter systems including the viscous Burgers equation. Another approach can treat infinite dimensional systems directly without any approximation. One example of this approach is based on the backstepping method (Tsubakino and Hara (2015), Tsubakino et al. (2012)). In (Krstic et al. (2008a)), full-state stabilization of the Burgers equation is discussed using the Hopf-Cole transformation and backstepping method. In a companion paper (Krstic et al. (2008b)), output feedback stabilization, trajectory generation, and tracking controlled problem are discussed with a method similar to (Krstic et al. (2008a)). While the viscous Burgers equation is transformed into a linear equation by the Hopf-Cole transformation, the cost function is transformed to a complicated nonlinear one. In (Vedantham (2000)), a simpler cost function which retains the behavior of the original one was proposed. As we can see from the above, there are many researches for stabilization of the nonlinear PDEs by state transformations. However, an optimal control problem including trajectory generation and tracking for nonlinear PDEs is not well studied, although it is needed for practical applications.

In this paper, we consider an optimal control problem for the viscous Burgers equation with Dirichlet boundary conditions. We treat a problem to control the Burgers equation such that it achieves the desired terminal state at the terminal time using the Hopf-Cole transformation and a state feedback. Optimal control generates a trajectory and a tracking controller with the best performance. Through the Hopf-Cole transformation, the optimal control problem for the viscous Burgers equation can be

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transformed into that for a linear heat equation. A state feedback is adopted to simplify the transformed boundary condition. Adopting the cost function for the transformed state, we can achieve the desired state without complicated calculations. Owing to those transformations the solution of the transformed heat equation can be calculated exactly. Furthermore the optimal control input for the heat equation is obtained as a solution of a Fredholm integral equation (Sakawa (1964)). We also discuss the class of achievable desired terminal states. Finally, we demonstrate the effectiveness of the proposed approach through a numerical simulation.

2. THE VISCOUS BURGERS EQUATION AND THE HOPF-COLE TRANSFORMATION

2.1 The Burgers equation and the boundary conditions

Consider the viscous Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \tag{1}$$

where $t \geq 0$ is a time variable, $x \in [0, 1]$ is a space variable, $u(x, t)$ is the state and $\nu > 0$ is the viscosity parameter. The initial condition and the boundary conditions are given by

$$u(x, 0) = u_0(x) \tag{2}$$

$$u(0, t) = 0, \quad u(1, t) = U(t) \tag{3}$$

where $U(t) \in \mathbb{R}$ is a control input. We assume $u_0(x) = 0$ in this paper for simplicity. Note that our method can be extended to any initial value of $u_0(x)$. Consider a problem to achieve the desired terminal state $u^*(x)$ at $t = T$ under the conditions (1), (2), and (3).

$$u(x, T) = u^*(x) \tag{4}$$

2.2 The Hopf-Cole transformation

For the Burgers equation, let us consider the following state transformation known as the Hopf-Cole transformation (Hopf (1950))

$$\varphi(x, t) = \exp\left(-\frac{1}{2\nu}\int_0^x u(y, t)\mathrm{d}y\right) \tag{5}$$

where $\varphi(x, t)$ is a new state variable. The inverse transformation of (5) becomes

$$u(x, t) = -2\nu\frac{\varphi_x(x, t)}{\varphi(x, t)}. \tag{6}$$

Substituting the transformation (6) into the system (1)-(3), we obtain the following equations.

$$\begin{cases} \varphi_t = \nu\varphi_{xx} + C(t)\varphi \\ \varphi_x(0, t) = 0, \quad \varphi_x(1, t) + \dfrac{U(t)}{2\nu}\varphi(1, t) = 0 \\ \varphi(x, 0) = 1 \end{cases} \tag{7}$$

Here, $C(t)$ is a constant of integration. To eliminate the term $C(t)$ in (7), the following transformation is used.

$$\hat{\varphi}(x, t) = \varphi(x, t)\exp\left(-\int_0^t C(\tau)\mathrm{d}\tau\right) \tag{8}$$

Using the above transformation, (7) is transformed into

$$\begin{cases} \hat{\varphi}_t = \nu\hat{\varphi}_{xx} \\ \hat{\varphi}_x(0, t) = 0, \quad \hat{\varphi}_x(1, t) + \dfrac{U(t)}{2\nu}\hat{\varphi}(1, t) = 0 \\ \hat{\varphi}(x, 0) = 1. \end{cases} \tag{9}$$

In order to solve (9) by the Laplace transform, let us consider a parallel shift as

$$\phi(x, t) = -1 + \hat{\varphi}(x, t). \tag{10}$$

Then we obtain the following equations

$$\begin{cases} \phi_t = \nu\phi_{xx} \\ \phi_x(0, t) = 0, \quad \phi_x(1, t) + \dfrac{U(t)}{2\nu}(\phi(1, t) + 1) = 0 \\ \phi(x, 0) = 0. \end{cases} \tag{11}$$

Note that the boundary condition at $x = 1$ is the Robin boundary condition. To simplify this boundary condition, we select the control input $U(t)$ as

$$U(t) = -2\nu\frac{V(t)}{1 + \phi(1, t)}. \tag{12}$$

Here $V(t) \in \mathbb{R}$ is a new control input. Then, the boundary condition at $x = 1$ becomes a simpler form.

$$\phi_x(1, t) = V(t) \tag{13}$$

Finally we obtain the following linear heat equation.

$$\begin{cases} \phi_t = \nu\phi_{xx} \\ \phi_x(0, t) = 0, \quad \phi_x(1, t) = V(t) \\ \phi(x, 0) = 0 \end{cases} \tag{14}$$

The desired terminal state $u^*(x)$ is transformed by (5), (8) and (10) as

$$\begin{cases} \phi^*(x) = -1 + \varphi^*(x)\exp\left(-\displaystyle\int_0^T C(\tau)\mathrm{d}\tau\right) \\ \varphi^*(x) = \exp\left(-\dfrac{1}{2\nu}\displaystyle\int_0^x u^*(y)\mathrm{d}y\right) \end{cases} \tag{15}$$

where $\phi^*(x)$ is the transformed desired terminal state. Since $\phi^*(x)$ includes the term $C(t)$, we need to evaluate the integration constant $C(t)$. However, as can be seen from (6), the inverse Hopf-Cole transformation is not injective. Due to this property, there are many terminal states $\phi^*(x)$ corresponding to the same terminal state $u^*(x)$ other than (15). One example of this terminal state $\phi^*(x)$ is given as

$$\begin{cases} \phi^*(x) \equiv \phi(x, T) = -1 + \varphi^*(x) \\ \varphi^*(x) = \exp\left(-\dfrac{1}{2\nu}\displaystyle\int_0^x u_1(y)\mathrm{d}y\right). \end{cases} \tag{16}$$

We can easily find that this state corresponds to $u^*(x)$ by using the inverse transformation of (5), (8) and (10). In this paper, we use (16) instead of (15) since the constant $C(t)$ does not appear in (16).

3. OPTIMAL CONTROL OF THE HEAT EQUATION

3.1 An exact solution to the heat equation

The general solution of (14) is obtained by using the Laplace transform. Transforming (14) yields

$$\begin{cases} s\Phi(x, s) = \nu\Phi_{xx}(x, s) \\ \Phi_x(0, s) = 0, \ \Phi_x(1, s) = V(s) \end{cases} \tag{17}$$