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Notions and Sufficient Conditions for Pointwise Asymptotic Stability in Hybrid Systems ⋆

Rafal K. Goebel * **Ricardo G. Sanfelice** **

* *Department of Mathematics and Statistics, Loyola University Chicago, 1032 W. Sheridan Road, Chicago, IL 60660, USA (e-mail: rgoebel1@luc.edu).*

** *Department of Computer Engineering, University of California, Santa Cruz, CA, 95064, USA (e-mail: ricardo@ucsc.edu).*

Abstract: Pointwise asymptotic stability is a property of a set of equilibria of a dynamical system, where every equilibrium is Lyapunov stable and every solution converges to some equilibrium. Hybrid systems are dynamical systems which combine continuous-time and discrete-time dynamics. In this paper, they are modeled by a combination of differential equations or inclusions, of difference equations or inclusions, and of constraints on the resulting motions. Sufficient conditions for pointwise asymptotic stability of a closed set are given in terms of set-valued Lyapunov functions: they require that the values of the Lyapunov function shrink along solutions. Cases of strict and weak decrease are considered. Lyapunov functions, not set-valued, which imply that solutions have finite length are used in sufficient conditions and related to the set-valued Lyapunov functions. Partial pointwise asymptotic stability is also addressed.

Keywords: Hybrid Systems, Stability, Lyapunov Methods, Dynamical Systems Techniques

1. INTRODUCTION

Hybrid dynamical systems exhibit features characteristic of continuous-time dynamical systems (flow) and features characteristic of discrete-time systems (jumps). Modeling of cyber-physical systems or mechanical systems with impacts, logic-based control algorithms, uncertainty and worst-case scenarios, etc. motivate the interest in hybrid systems in the control engineering and control theory literature. This paper models hybrid systems as hybrid inclusions, as in Goebel et al. (2009), Goebel et al. (2012).

Pointwise asymptotic stability (also known as semistability) is a property of a set of equilibria in a dynamical system, where every equilibrium is Lyapunov stable and from a neighborhood of it, every solution converges to possibly another equilibrium. Motivated by applications to consensus algorithms in Hui et al. (2008), hysteresis in Oh et al. (2009), and chemical processes and thermodynamics in Haddad et al. (2010), this stability concept has been analyzed in the setting of differential equations by Bhat and Bernstein (1999), Bhat and Bernstein (2003) and in the setting of differential inclusions by Hui et al. (2009). Bhat and Bernstein (2003) proposed a sufficient condition using a standard Lyapunov function and, additionally, a requirement that solutions don't flow in directions tangent to the set of equilibria. A standard Lyapunov function, on its own, cannot give a sufficient condition except the case of a single equilibrium. Sufficient conditions related to some results of this paper are in Bhat and Bernstein (2010), where arc-length-based Lyapunov sufficient conditions are proposed. For difference inclusions, necessary and sufficient conditions involving a set-valued Lyapunov function are in Goebel (2011), motivated by the idea of Moreau (2005): if the convex hull of positions of k agents decreases sufficiently along solutions of the system, then agents reach consensus. The sufficient condition of Moreau (2005) implies pointwise asymptotic stability of the set of consensus states, and naturally generalizes to sufficient conditions involving maps beyond the convex hull, as in Angeli and Bliman (2006), Goebel (2011). The use of general set-valued maps allows for converse Lyapunov results, i.e., necessary conditions, and enables characterizing robustness of pointwise asymptotic stability Goebel (2014). Some work on semistability for switching systems in Hui (2011) and for hybrid systems in Hui (2010) has appeared, but has not addressed necessary or sufficient conditions for pointwise asymptotic stability in the hybrid setting.

For a brief illustration of the set-valued approach, consider I agents, with states $x_i \in \mathbb{R}^l$ for $i = 1, 2, \ldots,$ who communicate and agree on a target a in the convex hull of x_i's, move toward a according to $\dot{x}_i = a - x_i$ for $T > 0$ amount of time, then communicate again, agree on a new a, and repeat. One can model this in the hybrid framework of Goebel et al. (2009), Goebel et al. (2012) by

$$\dot{x}_i = a - x_i, \ \dot{a} = 0, \ \dot{\tau} = -1 \quad \text{if } \tau \geq 0,$$

which describes the flow, and

$$x_i^+ = x_i, \ a^+ \in \text{con}\{x_1, x_2, \ldots, x_I\}, \ \tau^+ = T \quad \text{if } \tau = 0,$$

which describes the jumps. That the states x_i converge to the same point, and that states where $x_1 = x_2 = \cdots =$

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$x_I = a$ are Lyapunov stable, can be established by relying, for example, on the set-valued mapping given by

$$w(x,a) = \mathrm{con}\{x_1, x_2, \ldots, x_I, a\},$$

where $x = (x_1, x_2, \ldots, x_I)$. After a period of flow, $w(x,a)$ is a subset of the initial $w(x,a)$. Similarly, after a jump, $a^+ \in \mathrm{con}\{x_1, x_2, \ldots, x_I\}$, $x^+ = x$, and so $w(x^+, a^+) \subset w(x,a)$. These two weak decrease properties of w remain true even if, initially, a is not in the convex hull of x_i's. Based on this paper, and subject to verifying some basic properties of w, one can conclude stability. Furthermore, subject to verifying that $w(x,a)$ remains constant along flows and jumps only if $x_1 = x_2 = \cdots = x_I = a$, an invariance-based result applies and concludes consensus, i.e., convergence of x_i's to the same limit. Details are in Example 6.6, after the necessary theory is developed.

2. HYBRID INCLUSIONS

This paper considers hybrid systems modeled as hybrid inclusions as described below. For further details, consult Goebel et al. (2009), Goebel et al. (2012). Below, $C, D \subset \mathbb{R}^n$ are sets, called, respectively, the *flow set* and the *jump set* and $F, G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are set-valued mappings, called, respectively, the *flow map* and *jump map*. A hybrid system is represented by

$$\begin{aligned} x \in C \quad \dot{x} &\in F(x) \\ x \in D \quad x^+ &\in G(x). \end{aligned} \tag{1}$$

A special case of (1) is provided by systems where the flow and jump maps are functions, so that we have

$$\begin{cases} x \in C \quad \dot{x} = f(x) \\ x \in D \quad x^+ = g(x). \end{cases} \tag{2}$$

A set $E \subset \mathbb{R}^2$ is a *compact hybrid time domain* if

$$E = \bigcup_{j=0}^{J} I_j \times \{j\},$$

where $J \in \mathbb{N}$ and $I_j = [t_j, t_{j+1}]$, $j = 0, 1, \ldots, J$, for some $0 = t_0 \le t_1 \le t_2 \le \cdots \le t_{J+1}$. A set E is a *hybrid time domain* if, for each $(T, J) \in E$, the set $\{(t,j) \in E \,|\, t \le T, j \le J\}$ is a compact hybrid time domain. Equivalently, a hybrid time domain is a union of finitely or infinitely many intervals $[t_j, t_{j+1}] \times \{j\}$, where $0 = t_1 \le t_2 \le \ldots$, with the last interval, if it exists, possibly of the form $[t_j, t_{j+1})$ or $[t_j, \infty)$.

A function $\phi : E \to \mathbb{R}^n$ is a *solution to the hybrid system* (1) if E is a hybrid time domain, $\phi(0,0) \in \overline{C} \cup D$, and

- if $I^j := \{t \,|\, (t,j) \in E\}$ has nonempty interior, then $t \mapsto \phi(t,j)$ is locally absolutely continuous on I^j and
$$\phi(t,j) \in C \;\text{ for all } t \in \operatorname{int} I^j \;\text{ and}$$
$$\frac{d}{dt}\phi(t,j) \in F(\phi(t,j)) \;\text{ for almost all } t \in I^j;$$
- if $(t,j) \in E$ and $(t, j+1) \in E$ then
$$\phi(t,j) \in D \;\text{ and }\; \phi(t,j+1) \in G(\phi(t,j)).$$

In what follows, the domain of a solution ϕ (represented by E above) is denoted by $\operatorname{dom}\phi$. A solution $\phi : E \to \mathbb{R}^n$ is *maximal* if it cannot be extended, and *complete* if $\operatorname{dom}\phi$ is unbounded. Throughout the paper, the following stands.

Assumption 2.1. *Maximal solutions to* (1) *are complete.*

For conditions guaranteeing that maximal solutions are complete, see (Goebel et al., 2012, Proposition 2.10 and Proposition 6.10). In what follows, $\mathcal{S}$ denotes the set of all maximal, and hence complete, solutions to (1), $\mathcal{S}(x)$ denotes the set of maximal solutions to (1) that start from x, and for a set $K \subset \mathbb{R}^n$, $\mathcal{S}(K) := \bigcup_{x \in K} \mathcal{S}(x)$.

The set-valued analysis terms used below come from Rockafellar and Wets (1998). For further discussion, in the setting of hybrid systems, see Goebel et al. (2012).

Definition 2.2. *The hybrid system* (1) *satisfies the* hybrid basic assumptions *if its data,* (C, F, D, G)*, satisfies the following conditions: the sets* $C, D \subset \mathbb{R}^n$ *are closed; the set-valued mappings* $F, G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *are locally bounded and outer semicontinuous; for every* $x \in C$*,* $F(x)$ *is nonempty, closed, and convex; for every* $x \in D$*,* $G(x)$ *is nonempty and closed.*

In the special case of (2), the system satisfies the hybrid basic assumptions if $C, D \subset \mathbb{R}^n$ are closed and $f : C \to \mathbb{R}^n$, $g : D \to \mathbb{R}^n$ are continuous functions.

3. POINTWISE ASYMPTOTIC STABILITY

The definition below is global, in the sense that convergence of all solutions to (1) is required. The local case can be dealt with, but is not considered in this paper.

Definition 3.1. *A set* $A \subset \mathbb{R}^n$ *is* pointwise asymptotically stable *if*

(a) every $a \in A$ *is* Lyapunov stable*, that is, for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that every solution* ϕ *to* (1) *with* $\|\phi(0,0) - a\| < \delta$ *satisfies* $\|\phi(t,j) - a\| < \varepsilon$ *for every* $(t,j) \in \operatorname{dom}\phi$*, and*
(b) for every solution ϕ *to* (1)*,* $\lim_{t+j\to\infty} \phi(t,j)$ *exists and is contained in* A*.*

3.1 Structural properties of solutions

In absence of pointwise asymptotic stability, or a similar property, limits of solutions to a hybrid system (and, in fact, to a simple differential equation), even if they exist, may depend irregularly on initial conditions. For example, for $\dot{x} = -x(x-1)^2$, limits equal 0 for solutions from $(-\infty, 1)$ and equal 1 otherwise. Limits of solutions depend discontinuously on initial conditions at $x = 1$. Note that the smallest globally asymptotically stable set here is $[0,1]$, and the discontinuity occurs at a point in A. In fact, the limits may depend discontinuously on initial conditions even when the hybrid inclusion has linear flow and jump maps, due to the geometry of flow and jump sets.

As described below, pointwise asymptotic stability leads to reasonable dependence of solutions and their limits on initial conditions, and then to regularity of reachable sets etc. For an exposition of graphical convergence of hybrid arcs, featured in the next result, see Goebel et al. (2012).

Theorem 3.2. *Suppose that* (1) *satisfies the hybrid basic assumptions. For every sequence* $\phi_i \in \mathcal{S}$ *with convergent* $\phi_i(0,0)$*, there exists a graphically convergent subsequence, which is not relabeled, such that*