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On the Formation Rejoin Problem ⋆

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Abstract: We consider the problem of a maneuvering vehicle performing a rejoin to close formation with another maneuvering vehicle. The dynamics of relative position and velocity is shown to be governed by a linear time varying dynamics. Using *rotationally invariant* potential functions to exploit the symmetry present in the dynamics, we are able to construct Lyapunov functions that, along with their time derivative, are independent of time allowing us to conclude, in the case of linear feedback, exponential stability of the time varying closed loop system. The situation with nonlinear feedback is somewhat more delicate, requiring the use of Matrosov's theorem to prove uniform global asymptotic stability. Performance of the approach is illustrated by rejoining to an aggressively maneuvering flight leader using a saturating control law.

Keywords: Maneuvering vehicles. Stability. Matrosov theorem. Rotational symmetry.

1. INTRODUCTION

We are interested in the problem of a maneuvering vehicle performing a rejoin to close formation with another maneuevering vehicle. Our interest stems from the first author's experience flying jets in the US Air Force. With appropriate training, one finds that keeping the desired position within the formation may be accomplished using only (visual) information providing relative position (velocity and acceleration) in the frame of the flight leader, even under conditions of rather aggressive maneuvering. Naturally, it is important that the flight leader ("lead") maneuver within an appropriate class of maneuvers but, as seen in airshows, formation keeping (as we used to say, welded wing) can be reasonably achieved under full aerobatic maneuvering. Lead is typically highly skilled (and experienced), capable of providing a very smooth platform for the wingman (or wingmen) to follow. In order to *be* in formation, one must bring the vehicles together, hence the notion of a formation rejoin. This too is achieved largely with (visual) reference to flight lead, that is, working in some sort of local coordinates. The use of (curvilinear) coordinate systems that are adapted to a task is well known in path following and maneuver regulation, see, e.g., Samson (1995), Hauser and Hindman (1995), Saccon et al. (2013).

These ideas are also motivated by situations where a maneuvering entity is providing local navigational services to a family of maneuvering vehicles. Consider, for instance, a group of underwater vehicles working together with a team of divers without access to often used navigation aids (such as GPS) and without high speed communications. Location services may then be provided by a constellation of surface vehicles that maneuver in support of the underwater mission (Abreu and Pascoal, 2015).

In this paper, we set out to show that these well known (to pilots) ideas can be explored mathematically and are, in a sense, not so surprising when viewed in a natural setting.

2. EQNS OF MOTION IN FLIGHT LEAD FRAME

The position and orientation of the flight leader frame (at time $t \in \mathbb{R}$) is given by $x_d(t) \in \mathbb{R}^3$ and $R_d(t) \in SO(3)$. The velocity and acceleration, $\dot{x}_d(t)$ and $\ddot{x}_d(t)$, are bounded for admissible lead trajectories. Furthermore, the angular velocity $\omega_d(t)$ satisfying

$$\dot{R}_d(t) = R_d(t)\,\widehat{\omega}_d(t) \tag{1}$$

is also bounded. Here, the *hat* operator takes a vector $\omega \in \mathbb{R}^3$ to the skew symmetric matrix

$$\widehat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

representing the cross product operator $\widehat{\omega}\, v = \omega \times v$.

Recall that the rotation matrix $R_d(t)$ describing the orientation of the flight lead frame maps local vectors (those expressed in that frame) to the spatial frame, $y_{spatial} = R_d(t)\, y_{local}$. Thus, the position of a maneuvering vehicle can be expressed in the flight lead frame as

$$p = R_d^T(t)\,[x - x_d(t)]$$

since the inverse of a rotation matrix is its transpose. In the same way, the velocity and acceleration vectors are given by

$$\begin{aligned} v &= R_d^T(t)\,[\dot{x} - \dot{x}_d(t)] \\ a &= R_d^T(t)\,[\ddot{x} - \ddot{x}_d(t)] \end{aligned}$$

in the flight lead frame. Using (1), we see that a vehicle maneuvering in the flight leader frame is described by the linear *time-varying* control system

$$\begin{aligned} \dot{p} &= v - \widehat{\omega}_d(t)\, p \\ \dot{v} &= a - \widehat{\omega}_d(t)\, v \end{aligned} \tag{2}$$

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with state (p, v) and input a, written equivalently as

$$\begin{bmatrix} \dot{p} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} -\widehat{\omega}_d(t) & I \\ 0 & -\widehat{\omega}_d(t) \end{bmatrix} \begin{bmatrix} p \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} a \,. \tag{3}$$

3. ENERGY CONSIDERATIONS

The time varying terms induced by the use of a rotating frame should not affect the energy in the system since they do no work. Indeed, taking $T = \frac{1}{2} v^T v$ to be some sort of kinetic energy of the system, we see that

$$\dot{T} = -v^T \widehat{\omega}_d(t)\, v + v^T a = v^T a$$

since $\widehat{\omega}_d(t)$ is skew symmetric. This kinetic energy expression is not directly influenced by the frame rotation described by $\widehat{\omega}_d(t)$.

What sort of *potential* would exhibit a similar property? Apparently, a pure quadratic will do the trick in much the same way as for the kinetic energy. Indeed, if we take the total energy to be

$$E = \tfrac{1}{2} k_p p^T p + \tfrac{1}{2} v^T v \tag{4}$$

with k_p a positive scalar, we find that

$$\dot{E} = k_p p^T (-\widehat{\omega}_d(t)\, p + v) + v^T a = k_p p^T v + v^T a \tag{5}$$

which is again independent of $\widehat{\omega}_d(t)$.

What is special about the potential $U(p) = \frac{1}{2} k_p p^T p$ that leads to $DU(p) \cdot \widehat{\omega}\, p = 0$? Can this be generalized to a larger class of potential (energy) functions? The key here is symmetry: the potential $U(p)$ is *rotationally invariant* in that $U(Rp) = U(p)$ for all $p \in \mathbb{R}^3$ and $R \in SO(3)$. We have the following lemma.

Lemma 1. $U : \mathbb{R}^3 \to \mathbb{R}$ is rotationally invariant if and only if

$$DU(p) \cdot \widehat{\omega}\, p = 0, \quad \forall\, p, \omega \in \mathbb{R}^3 \,. \tag{6}$$

Proof. ($\Rightarrow$) Let p, $\omega \in \mathbb{R}^3$ be arbitrary and set $R(t) = e^{t\widehat{\omega}} \in SO(3)$, $t \in \mathbb{R}$. Then, since $U(R(t)p) = U(p)$, $t \in \mathbb{R}$, by rotational invariance, we see that

$$0 = \frac{d}{dt}\{U(R(t)p)\} = DU(R(t)p) \cdot \widehat{\omega}\, R(t)\, p$$

giving (6) at $t = 0$.

($\Leftarrow$) Let $\bar{R} \in SO(3)$ and $p \in \mathbb{R}^3$ be arbitrary, choose $\omega \in \mathbb{R}^3$ such that $\bar{R} = e^{\widehat{\omega}}$ and define $R(t) = e^{t\widehat{\omega}}$, $t \in \mathbb{R}$, so that $R(1) = \bar{R}$. Then

$$U(R(t)p) = U(p) + \int_0^1 DU(R(\tau)p) \cdot \widehat{\omega}\, R(\tau)\, p \; d\tau = F(p)$$

since, by (6), the integral is zero for every $\tau \in \mathbb{R}$. □

The level sets of a rotationally invariant function are spherical so that

Corollary 2. $\nabla U(p) \parallel p$ and $DU(p) \cdot p > 0$ (away from $p = 0$) for an increasing rotationally invariant function $U(p)$.

Summarizing, we see that if the energy of the system is taken to be the sum of rotationally invariant kinetic and potential energies

$$E(p, v) = T(v) + U(p) \,,$$

it will evolve independently of $\omega_d(t)$ according to

$$\dot{E} = DU(p) \cdot v + DT(v) \cdot a \,.$$

4. FORMATION REJOIN BY STABILIZATION

Formation rejoin is accomplished using a trajectory that takes (p, v) to the origin asymptotically, and formation keeping is achieved by keeping (p, v) at the origin or at least small. Some type of stabilizing controller may be enlisted to do this task.

Now, aside from the time varying terms induced by the rotating frame, this system looks largely like a double integrator system, suggesting the use of a Proportional-Derivative control law of the form

$$a = -K_p p - K_v v \tag{7}$$

where, for instance, K_p and K_v are symmetric and positive definite, to stabilize the system providing both rejoin and formation keeping.

Now, a necessary condition that (7) be stabilizing for a time-varying $\omega_d(t)$ is that it be stabilizing for every time-invariant ω_d in the desired class. Recall that, when $\omega_d \equiv 0$, the suggested PD control law (7) does provide exponential stability for the closed loop system. Unfortunately, the PD control law does not in general provide stability. Indeed, consider a constant turning maneuver in the XY plane with desired angular velocity $\omega_d = [\,0 \;\, 0 \;\, 1\,]^T$ and feedback gains $K_v = I$ and $K_p = \mathrm{diag}([\,a \;\, b \;\, 1\,])$, with $a, b > 0$. For initial conditions in horizontal plane, the system evolves within that plane as $\dot{x} = Ax$ with

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -a & 0 & -1 & 1 \\ 0 & -b & -1 & -1 \end{bmatrix} \tag{8}$$

giving a characteristic polynomial of

$$\chi(s) = s^4 + 2s^3 + (a+b+3)s^2 + (a+b+2)s + ab - a - b + 2$$

so that the system is unstable if $a + b > ab + 2$, for instance, $a = 4$ and $b = 1/2$. Thus, the presence of rotation can indeed destabilize an otherwise OK control law! Note that, in this example, the potential $U(p) = \frac{1}{2} p^T K_p p$ giving rise to the proportional feedback $-K_p p = -\nabla U(p)$ is *not* rotationally invariant.

Rotational invariance of the energy thus seems like a good idea if we are looking to achieve various forms of stability and invariance. Indeed, if we have a rotationally invariant potential $U(p)$ that is positive definite and we use a control law of the form

$$a = -\nabla U(p) - \bar{k}_v(v) \tag{9}$$

then the energy

$$E(p, v) = U(p) + \tfrac{1}{2} v^T v \tag{10}$$

evolves according to

$$\dot{E} = -v^T \bar{k}_v(v) \,. \tag{11}$$

If $\bar{k}(v) \equiv 0$ then the energy is conserved, and if $v^T \bar{k}_v(v) > 0$ for nonzero v then the energy will dissipate over time. In either case, the closed loop system will be *uniformly stable* on bounded sublevel sets of the energy.

These sets are defined independently of time and their positive invariance (and decrease) is also described in a time independent fashion, in spite of the fact that the system dynamics *does* depend on time. Our job is to determine when and what type of convergence may be achieved under various choices of velocity feedback and potential energy (giving the position feedback).