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IFAC-PapersOnLine 49-18 (2016) 316–320

A New Kind of Linear-Quadratic Leader-Follower Stochastic Differential Game ⋆

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Abstract: This paper studies a kind of linear-quadratic leader-follower stochastic differential game, where we assume that the leader's information is a sub-σ-algebra of the follower's. By the maximum principle and stochastic filtering, the feedback Stackelberg equilibrium is derived.

Keywords: leader-follower stochastic differential game; linear-quadratic control; asymmetric information; filtering; feedback Stackelberg equilibrium.

1. INTRODUCTION

We consider a finite time duration $T > 0$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, on which a standard Brownian motion $\{W_t, \widetilde{W}_t\}_{0 \le t \le T}$ with value in $\mathbb{R}^2$ is defined. Let $\{\mathcal{F}_t\}_{0 \le t \le T}$ be the natural filtration generated by $(W, \widetilde{W})$, $\mathcal{F}_T = \mathcal{F}$ and $\mathcal{G}_t \subseteq \mathcal{F}_t$.

Let the state $x^{u,w} \in \mathbb{R}$ satisfy the linear *stochastic differential equation* (SDE)

$$\begin{cases} dx_t^{u,w} = \big[Ax_t^{u,w} + B_1 u_t + B_2 w_t\big]dt \\ \qquad\qquad + Cx_t^{u,w}dW_t + \widetilde{C}x_t^{u,w}d\widetilde{W}_t, \\ x_0^{u,w} = x_0. \end{cases} \tag{1}$$

Here u is the follower's control, w is the leader's control, and both of them take values in $\mathbb{R}$; A, B_1, B_2, C and $\widetilde{C}$ are constants. In the first step of our game problem, for any chosen w, the follower chooses an $\mathcal{F}_t$-adapted, square-integrable control u^* to minimize his cost functional

$$\begin{aligned} J_1(u,w) =& \frac{1}{2}\mathbb{E}\Big[\int_0^T \big(Q_1|x_t^{u,w}|^2 + N_1|u_t|^2\big)dt \\ &+ G_1|x_T^{u,w}|^2\Big]. \end{aligned} \tag{2}$$

In the second step, knowing that the follower would take u^*, the leader wishes to choose a $\mathcal{G}_t$-adapted, square-integrable control w^* to minimize

$$\begin{aligned} J_2(u^*,w) =& \frac{1}{2}\mathbb{E}\Big[\int_0^T \big(Q_2|x_t^{u^*,w}|^2 + N_2|w_t|^2\big)dt \\ &+ G_2|x_T^{u^*,w}|^2\Big], \end{aligned} \tag{3}$$

where $Q_1, Q_2, G_1, G_2 \ge 0, N_1, N_2 \ne 0$ are constants. We refer to the above problem as a *linear-quadratic* (LQ) leader-follower stochastic differential game with asymmetric information. If there exists a control process pair (u^*, w^*), we refer to it as a Stackelberg equilibrium.

This paper is a counterpart of Shi, Wang and Xiong (2016), where another kind of LQ leader-follower differential game with asymmetric information is studied. The difference of the model in this paper lies in that the leader's information is a sub-σ-algebra of the follower's. This can been explained from many practical backgrounds. For example, in the newsvendors problem in the market for some product, the detailer plays a role of the follower and the manufacturer plays a role of the leader. The detailer faces many users and consumers, and then his information may be more than the manufacturer's since he could keep some information in the market as secrets which are not known by the manufacturer. This motivates us to study the game problem in this paper. Other applications of our game model can be seen in principal-agent problems (Cvitanić and Zhang (2013)) and stochastic Stackelberg differential games (Bagchi and Basar (1981); Yong (2002); Øksendal et al. (2013); Bensoussan et al. (2015)).

The rest of this paper is organized as follows. In Section 2, we present the main result. The game problem consists of an LQ optimal control problem of SDE with complete information for the follower, and an LQ optimal control problem of forward-backward stochastic differential equation (FBSDE) with partial information for the leader. The feedback Stackelberg equilibrium is characterized in terms of the forward-backward stochastic differential filtering equation and a system of Riccati equations. Finally, Section 3 gives concluding remarks.

⋆ Shi acknowledges the financial support from the National Natural Science Funds of China (11301011, 11201264, 11571205) and the Natural Science Fund for Distinguished Young Scholars of Shandong Province of China (JQ201401). Wang acknowledges the financial support from the National Natural Science Fund for Excellent Young Scholars of China (61422305), the National Natural Science Fund of China (11371228), the Natural Science Fund for Distinguished Young Scholars of Shandong Province of China (JQ201418), and the Research Fund for the Taishan Scholar Project of Shandong Province of China.

10.1016/j.ifacol.2016.10.184

2. MAIN RESULT

In this section, we will use the following two subsections to deal with the follower's and the leader's problem, respectively.

2.1 The Follower's Problem

Define the follower's Hamiltonian function as

$$H_1\big(t,x,u,w,q,k,\widetilde{k}\big)=q(Ax+B_1u+B_2w)-\frac{1}{2}Q_1x^2+kCx+\widetilde{k}\widetilde{C}x-\frac{1}{2}N_1u^2. \tag{4}$$

For any given w, by the LQ stochastic control theory (see Yong and Zhou (1999) or Tang (2003)), there exists an $\mathcal{F}_t$-adapted optimal control u^*, satisfying

$$0=N_1u_t^*-B_1q_t, \tag{5}$$

where $(q,k,\widetilde{k})\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ satisfies the *backward stochastic differential equation* (BSDE)

$$\begin{cases}-dq_t=\big(Aq_t+Ck_t+\widetilde{C}\widetilde{k}_t-Q_1x_t^{u^*,w}\big)dt\\ \qquad\quad -k_tdW_t-\widetilde{k}_td\widetilde{W}_t,\\ q_T=-G_1x_T^{u^*,w},\end{cases} \tag{6}$$

which is $\mathcal{F}_t$-adapted. Here $x^{u^*,w}$ denotes the optimal state of the follower.

Noting the terminal condition of (6) and the appearance of w, we let

$$q_t=-P_tx_t^{u^*,w}-\varphi_t,\ t\in[0,T]. \tag{7}$$

Here P_t is a deterministic and differentiable function with value in $\mathbb{R}$; (φ,β,γ) is an $\mathbb{R}^3$-valued, $\mathcal{F}_t$-adapted process, which satisfies the BSDE

$$\begin{cases}-d\varphi_t=\alpha_tdt-\beta_tdW_t-\gamma_td\widetilde{W}_t,\\ \varphi_T=0.\end{cases} \tag{8}$$

In the above, $\alpha\in\mathbb{R}$ is an $\mathcal{F}_t$-adapted process to be determined later.

Now, applying Itô's formula to (7), we arrive at

$$\begin{aligned}dq_t=&\big(-\dot{P}_tx_t^{u^*,w}-AP_tx_t^{u^*,w}-B_1P_tu_t^*\\&-B_2P_tw_t+\alpha_t\big)dt-\big(CP_tx_t^{u^*,w}+\beta_t\big)dW_t\\&-\big(\widetilde{C}P_tx_t^{u^*,w}+\gamma_t\big)d\widetilde{W}_t.\end{aligned} \tag{9}$$

Comparing (9) with (6), we have

$$k_t=-CP_tx_t^{u^*,w}-\beta_t, \tag{10}$$

$$\widetilde{k}_t=-\widetilde{C}P_tx_t^{u^*,w}-\gamma_t, \tag{11}$$

and

$$\begin{aligned}\alpha_t=&\big(\dot{P}_t+2AP_t+Q_1\big)x_t^{u^*,w}+B_1P_tu_t^*\\&+B_2P_tw_t+A\varphi_t-Ck_t-\widetilde{C}\widetilde{k}_t,\end{aligned} \tag{12}$$

respectively. By (5) and (7), we get

$$u_t^*=-N_1^{-1}B_1\big(P_tx_t^{u^*,w}+\varphi_t\big). \tag{13}$$

Substituting (13) into (12), we can obtain that if

$$\begin{cases}\dot{P}_t+(2A+C^2+\widetilde{C}^2)P_t-B_1^2N_1^{-1}P_t^2+Q_1=0,\\ P_T=G_1\end{cases} \tag{14}$$

admits a unique differentiable solution P_t, then

$$\alpha_t=(A-B_1^2N_1^{-1}P_t)\varphi_t-C\beta_t-\widetilde{C}\gamma_t-P_tB_2w_t. \tag{15}$$

By virtue of the standard Riccati equation theory, (14) admits a unique solution $P_t\geq 0$. See, e.g., Yong and Zhou (1999) or Tang (2003) for more details about the existence and uniqueness of the solution to Riccati equations.

By (15), the BSDE (8) takes the form

$$\begin{cases}-d\varphi_t=\big[(A-B_1^2N_1^{-1}P_t)\varphi_t+C\beta_t+\widetilde{C}\gamma_t\\ \qquad\qquad +P_tB_2w_t\big]dt-\beta_tdW_t-\gamma_td\widetilde{W}_t,\\ \varphi_T=0.\end{cases} \tag{16}$$

We summarize the above in the following theorem.

Theorem 2.1 *Let P_t satisfy (14). For chosen w of the leader, u^* given by (13) is the feedback optimal control for the follower's problem, where $(x^{u^*,w},\varphi,\beta,\gamma)$ is the unique $\mathcal{F}_t$-adapted solution to*

$$\begin{cases}dx_t^{u^*,w}=\big[(A-B_1^2N_1^{-1}P_t)x_t^{u^*,w}-B_1^2N_1^{-1}\varphi_t\\ \qquad\qquad +B_2w_t\big]dt+Cx_t^{u^*,w}dW_t+\widetilde{C}x_t^{u^*,w}d\widetilde{W}_t,\\ -d\varphi_t=\big[(A-B_1^2N_1^{-1}P_t)\varphi_t+C\beta_t+\widetilde{C}\gamma_t\\ \qquad\qquad +B_2P_tw_t\big]dt-\beta_tdW_t-\gamma_td\widetilde{W}_t,\\ x_0^{u^*,w}=x_0,\ \varphi_T=0.\end{cases} \tag{17}$$

2.2 The Leader's Problem

Knowing that the follower will take u^* by (13), the state equation of the leader writes an FBSDE

$$\begin{cases}dx_t^w=\big[(A-B_1^2N_1^{-1}P_t)x_t^w-B_1^2N_1^{-1}\varphi_t\\ \qquad\qquad +B_2w_t\big]dt+Cx_t^wdW_t+\widetilde{C}x_t^wd\widetilde{W}_t,\\ -d\varphi_t=\big[(A-B_1^2N_1^{-1}P_t)\varphi_t+C\beta_t+\widetilde{C}\gamma_t\\ \qquad\qquad +B_2P_tw_t\big]dt-\beta_tdW_t-\gamma_td\widetilde{W}_t,\\ x_0^w=x_0,\ \varphi_T=0,\end{cases} \tag{18}$$

where we let $x^w\equiv x^{u^*,w}$.

The leader would like to choose a $\mathcal{G}_t$-adapted optimal control w^* to minimize his cost functional

$$J_2(w)=\frac{1}{2}\mathbb{E}\Big[\int_0^T\big(Q_2|x_t^w|^2+N_2|w_t|^2\big)dt+G_2|x_T^w|^2\Big]. \tag{19}$$

Different from the follower's problem, the leader faces a stochastic optimal control problem of FBSDE with partial information. See, e.g., Huang, Wang and Xiong (2009); Øksendal and Sulem (2009); Xiao and Wang (2011) for more details about this topic. See also Tang (1998); Wang, Wu and Xiong (2013, 2015) for late developments on optimal control of stochastic systems with noisy observations.

We define the leader's Hamiltonian function as

$$\begin{aligned}&H_2\big(t,x^w,w,\varphi,\beta,\gamma;y,z,\widetilde{z},p\big)\\&=y\big[(A-B_1^2N_1^{-1}P_t)x^w-B_1^2N_1^{-1}\varphi+B_2w\big]\\&\quad+p\big[(A-B_1^2N_1^{-1}P_t)\varphi+C\beta+\widetilde{C}\gamma+B_2P_tw\big]\\&\quad+zCx^w+\widetilde{z}\widetilde{C}x^w+\frac{1}{2}\big(Q_2|x^w|^2+N_2|w|^2\big).\end{aligned} \tag{20}$$

According to the maximum principle introduced in Xiao and Wang (2011), there exists a $\mathcal{G}_t$-adapted optimal control w^* satisfying

$$0=N_2w_t^*+B_2P_t\mathbb{E}[p_t|\mathcal{G}_t]+B_2\mathbb{E}[y_t|\mathcal{G}_t], \tag{21}$$