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Stochastic Stability via Lyapunov Functions without Differentiability at Supposed Equilibria ⋆

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Abstract: The objective of this paper is to tackle point stability of stochastic systems without requiring noise diffusion coefficients to vanish at the point of interest. Behavioral differences between vanishing coefficients and non-vanishing coefficients arising in adding stochastic noises are illustrated by examples. To identify the differences appropriately, this paper examines the concept of equilibria and proposes several stability properties by introducing the notions of instantaneous points and almost sure equilibria. In addition to clarifying the relationship between those stability properties, Lyapunov-type characterizations are presented as sufficient conditions for those stability properties by making use of functions which are not necessarily twice differentiable at the point of interest. Discussion is also given to relate the stability property of an instantaneous point with noise-to-state stability.

Keywords: Stochastic systems, Nonlinear control systems, Stability analysis, Lyapunov functions.

1. INTRODUCTION

During the last half century, stability analysis of stochastic dynamical systems has been extensively studied, e.g., Bardi and Cesaroni (2005); Deng et al. (2001); Florchinger (1993); Khasminskii (2012); Kozin (1969); Kushner (1967a,b); Liu et al. (2008); Yin et al. (2011). The study has led to the development of stochastic Lyapunov theory. One of unique points of stochastic stability theory is the variety of notions such as stability in probability, moment stability and almost sure asymptotic stability (see Khasminskii (2012); Mao (2007); Kozin (1969)). In most of the preceding studies on these stability notions, it is common to assume that the diffusion coefficients of the stochastic noise vanish at the origin of the state space, which means that the origin is an equilibrium. However, there are many stochastic systems whose origin in the state space is not an equilibrium. Indeed, adding noises to deterministic systems to model random oscillations or uncertainties, which often arise in practical modeling, renders the origin an non-equilibrium point. Another unique feature of stochastic stability theory is that even when the origin is an equilibrium, a lot of stochastic systems do not admit Lyapunov functions which are C^2 at the origin, although their differential equations consist of smooth functions. Such an example is given in Remark 5.5 of Khasminskii (2012).

When it comes to stability in probability of an equilibrium, according to Kushner (1967b), the existence of a continuous Lyapunov function is necessary, while the existence of a C^2 Lyapunov function is sufficient (e.g., Khasminskii (2012); Kushner (1967a)). This fact together with Remark 5.5 of Khasminskii (2012) hints that asking Lyapunov functions for being C^2 at the equilibrium is too demanding.

This paper tackles the issues of stability of a non-equilibrium origin and Lyapunov functions which are not C^2 at the origin. The reader may suspect that the two issues can be addressed independently. However, one of the main massages of this paper is that the two issues are coupled. This paper demonstrates that one issue can be addressed with the help of investigation of the other. Importantly, this coupling leads us to the keys, reinvestigation of properties of the origin and definition of several detailed properties of stability. In addition to the abovementioned developments, this paper also relates them to the notion of noise-to-state stability introduced by Deng et al. (2001), which is not a property of a point, but a system property. Finally, a tool to construct non-C^2 Lyapunov functions solving the above two issues from Lyapunov functions of deterministic systems is proposed in this paper. All the proofs are omitted due to space limitation. Some sketches or keys are given as far as space permits.

Notations. $\mathbb{R}^d$ is the d-dimensional Euclidean space, especially $\mathbb{R} := \mathbb{R}^1$. $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^n$. A continuous function $y : [0, \infty) \to [0, \infty)$ is said to be of class $\mathcal{K}$ if it is strictly increasing and $y(0) = 0$. A class $\mathcal{K}$ function y is said to be of $\mathcal{K}_\infty$ if $\lim_{s\to\infty} y(s) = \infty$. A continuous function $\mu : [0, \infty) \times [0, \infty) \to [0, \infty)$ is said to be of class $\mathcal{KL}$ if, for each fixed $t \in [0, \infty)$, the function $\mu(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed $s > 0$,

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$\mu(s,\cdot)$ is strictly decreasing and $\lim_{t\to\infty}\mu(s,t)=0$. For $a,b\in\mathbb{R}$, let $a\wedge b$ denote the minimum of a and b. Let $\mathcal{P}:=(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ be a filtered probability space, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra that is a collection of all the events, $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration of $\mathcal{F}$, and $\mathbb{P}$ is a probability measure. The probability and the expectation of some event A are written as $\mathbb{P}[A]$ and $\mathbb{E}[A]$, respectively. The vector-valued function $w:=[w_1,w_2,\ldots,w_d]^T\in\mathbb{R}^d$ is d-dimensional standard Brownian motion defined on $\mathcal{P}$. The differential form of Stratonovich integral of a function $\sigma_\alpha:\mathbb{R}^n\to\mathbb{R}^n$ in $w_\alpha(t)$ is denoted by $\sigma_\alpha(x)\circ dw_\alpha(t)$ for $\alpha=1,2,...,d$.

2. MOTIVATING EXAMPLES

Consider the stochastic differential equation of the form

$$dx(t)=-ax(t)dt+2bx(t)\circ dw(t) \tag{1}$$

with $a,b>0$. The origin $x=0\in\mathbb{R}$ is an equilibrium in the sense that $x(0)=0$ results in $x(t)=0$ for all $t>0$. The solution to (1) is $x(t)=x(0)\exp(-at+2bw(t))$ as explained in Remark 5.5 of Khasminskii (2012). Hence, the process $x(t)$ converges to the origin $x=0$ in an appropriate sense. Pick a function $V_2(x)=x^2$. Then $(\mathcal{L}V_2)(x)=2(-a+4b^2)x^2\geq 0$ holds[1] for $a\leq 4b^2$. Hence, $V_2(x)=x^2$ does not describe the convergence. For $V_p(x)=|x|^p$ with $p<a/(2b^2)$, one obtains

$$(\mathcal{L}V_p)(x)=p(-a+2b^2p)|x|^p<0,\ \forall x\in\mathbb{R}\setminus\{0\}. \tag{2}$$

This suggests that the use of small p allows $V_p(x)=|x|^p$ to verify the convergence of the process $x(t)$ to the origin $x=0$. However, small p renders $V(x)$ non-differentiable at $x=0$, although the definition of $\mathcal{L}V(x)$ requires V to be of class C^2. It is interesting to know what kind of a conclusion a function $V(x)$ can draw actually about stability when it is not C^2 at the origin.

Addressing the above question is not limited to investigation of the validity of employing a non C^2 function. Consider

$$dx(t)=-x(t)dt+1\circ dw(t), \tag{3}$$

where $x(t)$ and $w(t)$ are scalars. The choice $V_1(x)=|x|$ which is not of class C^2 at $x=0$ yields

$$(\mathcal{L}V_1)(x)=-|x|,\quad \forall x\in\mathbb{R}\setminus\{0\}. \tag{4}$$

Hence, one may expect that the origin $x=0$ is stable and attractive in some sense. However, $x=0$ is not an equilibrium, and $x(0)=0$ does not result in $x(t)\equiv 0$. Indeed, the solution to (3) with $x(0)=0$ can be expressed as

$$x(t)=e^{-t}\int_0^t e^s dw(s).$$

Thus, Itô isometry (e.g., Khasminskii (2012); Øksendal (2013); Mao (2007) leads us to

$$\mathbb{E}[|x(t)|^2]=e^{-2t}\int_0^t e^{2s}ds=\frac{1}{2}(1-e^{-2t}).$$

This yields $\mathbb{E}[|x(t)|^2]\to 1/2$ as $t\to\infty$. The variance of $x(t)$ stays away from zero and does not converge to zero, which seems to contradict the inference of stability and attractivity of $x=0$ from (4). Hence, the use of $V(x)$ which is not C^2 *at the origin* requires careful attention and it is important to be able to precisely identify stability properties a function $V(x)$ can conclude. Example (3) also reveals delicate issues of stability-like properties when stochastic noise prevents the origin from being an actual equilibrium.

[1] $\mathcal{L}$ is the infinitesimal operator corresponding to the Lie derivative for deterministic systems.

3. SDE AND PROPERTIES OF THE ORIGIN

Consider the Stratonovich-type stochastic differential equation (SDE)

$$dx(t)=f(x(t))dt+\sum_{\alpha=1}^{d}\sigma_\alpha(x(t))\circ dw_\alpha(t), \tag{5}$$

where the drift term $f:\mathbb{R}^n\to\mathbb{R}^n$ is supposed to be locally Lipschitz, and the diffusion coefficients $\sigma_\alpha:\mathbb{R}^n\to\mathbb{R}^n$ are assumed to be continuously differentiable. The initial state is supposed to be deterministic and denoted by $x(0)=x_0\in\mathbb{R}^n$. System (5) is time-invariant in the sense that f and σ_α are not function of t, and they are functions of x instead. Throughout this paper, $f(0)=0$ is assumed. In other words, the origin $x=0$ of the deterministic part $dx/dt=f(x)$ is assumed to be an equilibrium, i.e., $dx/dt=f(x)$ is satisfied by $x(t)\equiv 0$ ($x(t)=0$ for any $t\in[0,\infty)$). It is stressed that the origin $x=0$ remains an equilibrium of all sample paths of (5) if and only if $\sigma_\alpha(0)=0$ holds for all $\alpha=1,2,...,d$. Considering the case of $\sigma_\alpha(0)\neq 0$, the origin $x=0$ is only a "supposed" equilibrium of (5).

For deterministic systems, stability is a property of equilibria. A point in the state space is stable in the sense of Lyapunov (or asymptotically stable) only if it is an equilibrium. In contrast, for stochastic systems, expecting an equilibrium in investigating a stability-like property is not always appropriate, due to diversity of stochastic behavior. In order to be able to deal with such diversity, for a given stochastic process $x:[0,\infty)\to\mathbb{R}^n$ with $x(0)=0\in\mathbb{R}^n$ defined on a probability space, define the following random variables

$$e_{\neq}:=\inf\{t>0\,|\,x(t)\neq 0\} \tag{6}$$

$$e_0:=\inf\{t>0\,|\,x(t)=0\}. \tag{7}$$

Note that we write $e_{\neq}=\infty$ if $\{t>0\,|\,x(t)\neq 0\}=\emptyset$. In the same way, let $e_0=\infty$ if $\{t>0\,|\,x(t)=0\}=\emptyset$. We use the following terminology:

Definition 1. Let $x:[0,\infty)\to\mathbb{R}^n$ be the solution of the time-invariant system (5) with $x_0=0$. The origin $x=0$ of (5) is said to be an *instantaneous point* if $\mathbb{P}[e_{\neq}=0]=1$ holds. The origin $x=0$ is said to be a *trap* if $\mathbb{P}[e_0=0]=1$ holds. □

Being an instantaneous point means that x leaves the origin instantaneously, while being a trap means that x returns to the origin instantaneously. Interestingly, for the (one-dimensional) Brownian motion, i.e., $dx(t)=dw(t)$ with $n=1$ and $w(0)=0$, the origin is both an instantaneous point and a trap, which is known as Blumenthal's 0-1 law. Thus, the two notions introduced in Definition 1 do not exclude each other. Since $x=0$ of the Brownian motion is an instantaneous point, the assumption $f(0)=0$ yields the following:

Proposition 2. If $x=0$ is not an instantaneous point, then it is a trap.