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Controlled synchronization of mechanical systems with a unilateral constraint ⋆

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Abstract: This paper addresses the controlled synchronization problem of mechanical systems subjected to a geometric unilateral constraint as well as the design of a switching coupling law to obtain synchronization. To define the synchronization problem, we propose a distance function induced by the quotient metric, which is based on an equivalence relation using the impact map. A Lyapunov function is constructed to investigate the synchronization problem for two identical one-dimensional mechanical systems. Sufficient conditions for the individual systems and their controlled interaction are provided under which synchronization can be ensured. We present a (coupling) control law which ensures global synchronization, also in the presence of grazing trajectories and accumulation points (Zeno behavior). The results are illustrated using a numerical example.

Keywords: Synchronization, measure differential inclusions, unilateral constraints, Lyapunov stability, hybrid systems

1. INTRODUCTION

Synchronization of coupled dynamical systems leads to 'motion in unison' which is a fundamental phenomenon appearing in, for example, biological and engineering systems. The synchronization of chaotic oscillators, neural systems and mechanical systems described by *smooth* nonlinear systems has been studied extensively, see Pikovsky et al. (2001); Nijmeijer and Rodriguez-Angeles (2003); Arenas et al. (2008); Strogatz (2003) and references therein. Synchronization of *nonsmooth* systems has received significantly less attention and to the best of the authors knowledge, the problem of synchronization for unilaterally constrained mechanical systems has not yet been addressed.

In this paper, synchronization is analyzed for mechanical systems with geometric unilateral constraints, which occur generally if mechanical systems (such as, e.g., robots) interact with a rigid environment. The dynamics of these systems comprises impacts which induce velocity jumps, rendering the system dynamics of an impulsive, hybrid nature (Leine and van de Wouw (2008); Goebel et al. (2012); Michel and Hu (1999)). For unilaterally constrained mechanical systems, accumulation points of infinitely many impact events can generally be observed, which is known as Zeno-behavior. To describe the dynamics which includes such accumulation points, system models in terms of Measure Differential Inclusions (MDIs) are employed in Moreau (1988); Leine and van de Wouw (2008).

Because impacts of unilaterally constrained mechanical systems are a consequence of collisions and therefore are state-triggered events (i.e., occur at a certain position), they generally do not occur at the same time instants for nearby trajectories. Therefore, one expects a small time-mismatch of the impact time instants even for arbitrarily close initial conditions. During this time (mismatch) interval, a large Euclidean error is observed, cf. Biemond et al. (2013); Brogliato et al. (1997); Forni et al. (2013); Leine and van de Wouw (2008); Menini and Tornambè (2001). Hence, the Euclidean synchronization error dynamics is generally unstable in the sense of Lyapunov and existing synchronization results are not applicable to mechanical systems with unilateral position constraints. An exception is the synchronization between a mechanical system and an observer, in which the impacts of the observer state can be made to coincide with the impacts of the mechanical systems, as exploited in Baumann and Leine (2015).

Recently, focusing on the stability of jumping trajectories, the 'peaking phenomenon' has been addressed for hybrid systems in the framework of Goebel et al. (2012) by considering stability in terms of a novel distance function which takes the jump characteristics into account, cf. Biemond

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et al. (2013, 2015). This approach has been extended in Postoyan et al. (2015) towards incremental stability. These approaches, however, are not applicable if either the time between state jumps can be arbitrarily small (especially in Zeno events), or if jumps can occur where the pre- and post-jump states are arbitrary close to each other. Both phenomena are generally expected in unilaterally constrained mechanical systems, motivating the synchronisation problem under study, in which hybrid trajectories are expressed by measure differential inclusions.

We distinguish three main contributions. First, we construct a distance function for mechanical systems with multiple degrees of freedom and a single geometric unilateral constraint, therewith extending the distance function design in Schatzman (1998)). This distance function can be used to define when solutions are considered close to synchronization or when they are synchronized. The synchronization problem formulation, which we establish based on the presented distance function, is applicable to generic mechanical systems with a unilateral constraint. To the best of the authors knowledge, this formulation is the first that is applicable to state-triggered hybrid systems and does not resort to Poincaré maps. Second, Lyapunov arguments are used to investigate this synchronization problem for the one-dimensional case and provide conditions on the individual systems and their controlled interaction which guarantee that synchronization indeed occurs. In contrast to the hybrid systems in Biemond et al. (2013); Forni et al. (2013), impacts with arbitrary small velocity jumps can occur, which severely complicates the Lyapunov function design and analysis. Third, we design a control law to enforce controlled synchronization using non-impulsive forces generated by the interaction network. Finally, the results are illustrated with a numerical example.

2. MECHANICAL SYSTEMS WITH A SINGLE UNILATERAL CONSTRAINT

We consider an n-DOF (degrees of freedom) mechanical system subjected to a single frictionless geometric unilateral constraint. The state of the system is described by the generalized coordinates $\boldsymbol{q}(t) \in \mathbb{R}^n$ and velocities $\boldsymbol{u}(t) \in \mathbb{R}^n$. The non-impulsive dynamics is described by the kinematic equation and the equation of motion given by

$$\begin{aligned} \dot{\boldsymbol{q}} &= \boldsymbol{u}, \\ \boldsymbol{M}\dot{\boldsymbol{u}} - \boldsymbol{h}(\boldsymbol{q},\boldsymbol{u},\boldsymbol{\tau},t) &= \boldsymbol{w}\lambda, \end{aligned} \tag{1}$$

where $\boldsymbol{h}(\boldsymbol{q},\boldsymbol{u},\boldsymbol{\tau},t)$ is a function of the state $(\boldsymbol{q},\boldsymbol{u})$, the control inputs $\boldsymbol{\tau}$ and the time t explicitly. We will use the notation $(\boldsymbol{x},\boldsymbol{y}) = \left(\boldsymbol{x}^\mathsf{T}\ \boldsymbol{y}^\mathsf{T}\right)^\mathsf{T}$, where $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n$. The mass matrix $\boldsymbol{M} = \boldsymbol{M}^\mathsf{T} \succ \boldsymbol{0}$ is symmetric and assumed to be constant and positive definite. The motion of the system is restricted by a single scleronomic geometric unilateral constraint $g(\boldsymbol{q}) \geq 0$, where $g : \mathbb{R}^n \to \mathbb{R}$ is an affine function of $\boldsymbol{q}$. The constraint velocity $\gamma(\boldsymbol{u}) = \frac{\mathrm{d}g(\boldsymbol{q}(t))}{\mathrm{d}t} = \boldsymbol{w}^\mathsf{T}\boldsymbol{u}$ is the time derivative of the constraint distance g, where $\boldsymbol{w} = \left(\frac{\partial g}{\partial \boldsymbol{q}}\right)^\mathsf{T}$ is the associated generalized force direction.

The force law for the constraint force λ is described by the inequality complementarity condition, see Glocker (2001) (also referred to as Signorini's law):

$$0 \leq g(\boldsymbol{q}) \perp \lambda \geq 0, \tag{2}$$

where $a \perp b$ denotes $ab = 0$. The admissible set of states is $\mathcal{A} := \left\{(\boldsymbol{q},\boldsymbol{u}) \in \mathbb{R}^{2n} \,\middle|\, g(\boldsymbol{q}) \geq 0\right\}$. The boundary of $\mathcal{A}$ is partitioned as $\partial\mathcal{A} = \partial\mathcal{A}^+ \cup \partial\mathcal{A}^-$ with $\partial\mathcal{A}^+ := \left\{(\boldsymbol{q},\boldsymbol{u}) \in \mathbb{R}^{2n} \,\middle|\, g(\boldsymbol{q}) = 0,\ \gamma(\boldsymbol{q},\boldsymbol{u}) \geq 0\right\}$ and $\partial\mathcal{A}^- = \left\{(\boldsymbol{q},\boldsymbol{u}) \in \mathbb{R}^{2n} \,\middle|\, g(\boldsymbol{q}) = 0,\ \gamma(\boldsymbol{q},\boldsymbol{u}) < 0\right\}$. An impact is imminent if the state is in $\partial\mathcal{A}^-$ because an impact is required for the system to remain in the admissible set $\mathcal{A}$. The impulsive dynamics is described by the impact equation

$$\boldsymbol{M}(\boldsymbol{u}^+ - \boldsymbol{u}^-) = \boldsymbol{w}\Lambda, \tag{3}$$

where $\boldsymbol{u}^-(t) = \lim_{\tau\uparrow 0} \boldsymbol{u}(t+\tau)$ and $\boldsymbol{u}^+(t) = \lim_{\tau\downarrow 0} \boldsymbol{u}(t+\tau)$ are the pre- and post-impact velocities, respectively. The constraint impulse Λ is given by the generalized Newton's law (see Glocker (2001)) with coefficient of restitution $e \in [0,1]$:

$$g(\boldsymbol{q}) = 0: \qquad 0 \leq \Lambda \perp \boldsymbol{w}^\mathsf{T}(\boldsymbol{u}^+ + e\boldsymbol{u}^-) \geq 0. \tag{4}$$

We note that infinitely many impacts can occur in a finite time interval, known as Zeno behavior or the accumulation of impact time instants. Our desire to accommodate the modeling of such behaviors motivates describing the dynamics with measure differential inclusions (1)–(4), which can be written in the compact form (see Moreau (1988); Leine and van de Wouw (2008))

$$\begin{aligned} &\mathrm{d}\boldsymbol{q} = \boldsymbol{u}\mathrm{d}t, \\ &\boldsymbol{M}\mathrm{d}\boldsymbol{u} - \boldsymbol{h}(\boldsymbol{q},\boldsymbol{u},\boldsymbol{\tau},t)\mathrm{d}t = \boldsymbol{w}(\lambda\mathrm{d}t + \Lambda\mathrm{d}\eta), \end{aligned}$$

with λ and Λ satisfying (2) and (4). The generalized coordinates $\boldsymbol{q} : \mathbb{R} \to \mathbb{R}^n$ are absolutely continuous functions in time and their measure $\mathrm{d}\boldsymbol{q}$ has density $\boldsymbol{u}$ with respect to the Lebesgue measure $\mathrm{d}t$. The generalized velocities $\boldsymbol{u} : \mathbb{R} \to \mathbb{R}^n$ are discontinuous due to the impulsive dynamics, but they are assumed to be functions of special locally bounded variation (see Ambrosio et al. (2000)), such that the pre- and post-impact velocities $\boldsymbol{u}^-(t)$ and $\boldsymbol{u}^+(t)$, respectively, are defined for every point in time. The measure $\mathrm{d}\boldsymbol{u}$ has a density $\dot{\boldsymbol{u}}$ with respect to the Lebesgue measure $\mathrm{d}t$ and a density $(\boldsymbol{u}^+ - \boldsymbol{u}^-)$ with respect to the atomic measure $\mathrm{d}\eta$, i.e., $\mathrm{d}\boldsymbol{u} = \dot{\boldsymbol{u}}\mathrm{d}t + (\boldsymbol{u}^+ - \boldsymbol{u}^-)\mathrm{d}\eta$. The atomic measure $\mathrm{d}\eta = \sum_i \mathrm{d}\delta_{t_i}$ is the sum of Dirac point measures $\mathrm{d}\delta_{t_i}$ at the discontinuity points t_i, cf. Glocker (2001).

As shown in Leine and Baumann (2014), the impact equation (3) together with the impact law (4) results in an explicit impact map $\bar{Z} : (\boldsymbol{q},\boldsymbol{u}^-) \mapsto (\boldsymbol{q},\boldsymbol{u}^+) = \bar{Z}\left(\boldsymbol{q},\boldsymbol{u}^-\right)$, where

$$\begin{aligned} &\bar{Z}\left(\boldsymbol{q},\boldsymbol{u}^-\right) = \left(\boldsymbol{q}, Z_q(\boldsymbol{u}^-)\right) \\ &\quad \text{with } Z_q(\boldsymbol{u}^-) = (1+e)\,\mathrm{prox}^{\boldsymbol{M}}_{\mathcal{T}_C(\boldsymbol{q})}(\boldsymbol{u}^-) - e\boldsymbol{u}^-, \\ &\quad \text{where } \mathcal{T}_C(\boldsymbol{q}) = \begin{cases} \left\{\boldsymbol{u} \middle| \boldsymbol{w}^\mathsf{T}\boldsymbol{u} \geq 0\right\} & \text{if } g(\boldsymbol{q}) = 0, \\ \mathbb{R}^n & \text{if } g(\boldsymbol{q}) > 0 \end{cases} \end{aligned} \tag{5}$$

and $\mathrm{prox}^{\boldsymbol{M}}_{\mathcal{T}}(\boldsymbol{u})$ denoting $\arg\min_{\boldsymbol{v}\in\mathcal{T}} \|\boldsymbol{u} - \boldsymbol{v}\|_{\boldsymbol{M}}$. In the following section, we consider the synchronization problem for mechanical systems of the form (1)–(4). The 'peaking phenomenon', which appears when the Euclidean synchronization error is considered, is induced by the nature of the underlying system. We construct a function d that takes the role of distance and is continuous when evaluated along solutions by explicitly incorporating the impact map $\bar{Z}$. The property of non-expansivity of $\bar{Z}$ as defined in Baumann and Leine (2015) leads to a great simplification in the construction of the distance function.