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A Hybrid Model of Opinion Dynamics with Limited Confidence

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Abstract: This paper proposes a novel model of opinion dynamics with limited confidence. As opposed to other models, the onset of reciprocal influences between individuals is not defined by absolute thresholds, but by a notion of relative relevance for each link. The proposed dynamics takes the form of a hybrid dynamical system, where topology and opinions are coupled but as distinct variables. We prove converge to opinion distributions where disconnected clusters of individuals agree locally, while each cluster holds a different opinion. We also illustrate the dynamics via simulations.

Keywords: Hybrid dynamical systems, opinion dynamics, Lyapunov

1. INTRODUCTION

Prior expertise in consensus and consensus-seeking dynamics is leading control scientists to study the evolution of opinions in social networks, which can be described by similar dynamics. Popular mathematical models have been previously proposed and studied in social sciences by Friedkin (2006), in physics by Castellano et al. (2009) and in applied mathematics by Lorenz (2006); Como and Fagnani (2011): an insightful historical perspective on this body of research can be found in Friedkin (2015). By now, the topic of opinion dynamics has become a popular and distinct topic in systems and control, as evidenced by papers like Acemoglu et al. (2013); Altafini (2013); Ravazzi et al. (2015); Parsegov et al. (2015).

Opinion-dependent limitations in the relative influence between individuals are a fundamental ingredient in several models of opinion dynamics. Their interest stems from their ability to prevent consensus in the long run, notwithstanding the attractive forces that drive opinions of connected individuals towards each other. Broadly intended, these limitations postulate that individuals do not influence each other if their opinions are too far apart. The simplest form of limitation –referred to as "bounded confidence"– is based on a fixed threshold: individuals interact if their opinions are closer than the threshold. The resulting nonlinear dynamics has been popularized in the control community by Blondel et al. (2009), who analyzed the seminal discrete-time model by Hegselmann and Krause (2002). Afterwards, many variations of this model have been studied, including continuous-time dynamics by Blondel et al. (2010), heterogeneous thresholds by Mirtabatabaei and Bullo (2012), continuous distributions of opinions by Roozbehani et al. (2008); Canuto et al. (2012), and multidimensional opinions by Nedic and Touri (2012); Bhattacharyya et al. (2013); Etesami and Başar (2015).

This paper proposes and studies a novel model of opinion dynamics with limited confidence, where reciprocal influences are not defined by absolute thresholds, but by a relative notion of relevance for each link. For this dynamics, the limit points are opinion distributions such that disconnected clusters of individuals agree locally, while each cluster holds a different opinion. Similar limit behaviors are shown by other limited confidence models as Hegselmann and Krause (2002). As our main result, we are able to prove uniform global asymptotic stability of the set of equilibria (and thus, convergence of solutions to such a set). For brevity, this paper does not contain the complete proof of this statement, which will be made available in an extended version. However, we do include some simulations that illustrate the dynamics and the role of the model parameters.

We want to emphasize that our dynamics is defined within a hybrid systems framework, which brings two important advantages. Firstly, we are able to build upon the well-established and comprehensive theory presented in Goebel et al. (2012). This theory is useful to guarantee existence and completeness of solutions, which are tricky for some non-hybrid models: for detailed discussions on this matter, see Ceragioli and Frasca (2012, 2015). At the same time, it allows us to prove stability and attractivity of the equilibria via a transparent Lyapunov argument. Secondly, the hybrid framework allows for considering the network topology as an independent (discrete) variable that interacts with the (continuous) opinion variable. To the best of our knowledge, previous works simply assume the topology to be a function of the current opinion, thus not allowing for memory or hysteresis effects. On the contrary, their inclusion is natural in the hybrid framework. In conclusion, the hybrid framework seems to us the ideal tool to study the complex co-evolutions of opinions and network structure that are inherent to real

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social networks. This paper contains a first example of this potentially rich and fruitful class of models.

The paper is organized as follows. Section 2 describes the model of opinion dynamics under consideration. Following a hybrid modeling, the jump and flow equations together with the flow and jump sets are precisely defined. In section 3, the main results in terms of convergence are proposed. Section 4 illustrates the evolution of the proposed dynamics and the role of the different parameters. Finally, a conclusion ends the paper.

2. THE MODEL

Consider n agents indexed in a set $i \in \mathcal{I} = \{1, \ldots, n\}$, each of them holding a time-dependent opinion $y_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$. Consider also a time-varying interaction pattern where for any pair $(h, k) \in \mathcal{I} \times \mathcal{I}$, such that $h \neq k$, agents h and k share opinions if $a_{hk} = a_{kh} \in \{0, 1\}$ is set to 1. We call the binary values a_{hk} edges, and they are defined for all indices (h, k) taking values in the index set:

$$\mathcal{E} := \{(i, j) : i \in \mathcal{I},\ j \in \mathcal{I} \setminus \{i\}\}.$$

Since we are considering symmetric interaction dynamics, namely $a_{hk} = a_{kh}$ for all $(h, k) \in \mathcal{E}$, set $\mathcal{E}$ above is redundant and it is convenient to introduce the reduced set

$$\mathcal{E}^+ := \{(i, j) : i, j \in \mathcal{I},\ j > i\}.$$

Based on the above reduced index set, we can define a vector

$$a := (a_{12}, a_{13}, \ldots, a_{1n}, a_{21}, \ldots, a_{n-2,n-1}, a_{n-2,n}, a_{n-1,n})$$

where clearly $a \in \{0, 1\}^{\frac{n(n-1)}{2}}$. Then all possible pairwise interactions among the n agents are described by the elements of a. In the sequel we will refer to elements of a interchangeably using the two notations $a_{hk} = a_{kh}$ whose meaning is not ambiguous as long as $h \neq k$. Based on the time-varying edges represented in a, each agent may have a variable number of active connections with other agents. It is then convenient to define the (augmented) degree of agent i, for each $i \in \mathcal{I}$, as

$$d_i = 1 + \sum_{j \neq i} a_{ij}.$$

Note that with this convention, the degree of any node is always strictly positive.

The model proposed in this paper aims at regulating both the continuous evolution of the agents' opinions (described by suitable variations of state y), and the discrete variations in the interaction pattern (described by instantaneous jumps of state a). Since the proposed model involves both continuous variations and instantaneous jumps of the state, we will adopt a hybrid framework for its description and analysis. More precisely, the overall state $x := (y, a)$ evolves in the following set:

$$(y, a) \in \mathbb{X} := \mathbb{R}^n \times \{0, 1\}^{\frac{n(n-1)}{2}}. \tag{1}$$

Regarding the dynamics of the overall model, we will consider the following flow equation for the overall state variable (y, a):

$$\begin{cases} \dot{y}_i = \displaystyle\sum_{j \in \mathcal{I} \setminus \{i\}} \frac{a_{ij}}{d_i d_j}(y_j - y_i) & \text{for all } i \in \mathcal{I} \\ \dot{a}_{ij} = 0 & \text{for all } (i, j) \in \mathcal{E}^+, \end{cases} \tag{2}$$

which is motivated by the fact that interactions only occur between pairs (i, j) of agents having an active link ($a_{ij} = 1$) and that the interaction is stronger if the two agents have a small number of active links (small values of d_i and d_j).

Note that along the flow dynamics (2), the connection graph remains constant ($\dot{a}_{ij} = 0$) during flowing of the hybrid solutions. Indeed, the change of topology of the interconnection graph is captured by a jump of the hybrid solution that leaves the opinions y unchanged and only affects the elements a_{ij} of a by the following set of jump rules that must be applied to each $(h, k) \in \mathcal{E}^+$:

$$\begin{cases} y_i^+ = y_i & \text{for all } i \in \mathcal{I} \\ a_{hk}^+ = 1 - a_{hk} \\ a_{ij}^+ = a_{ij} & \text{for all } (i, j) \in \mathcal{E}^+ \setminus \{(h, k)\} \end{cases} \quad (y, a) \in D_{hk}. \tag{3}$$

According to the above equation, a jump (toggle between 0 and 1) of edge a_{hk} is enabled when the state (y, a) belongs to the set

$$D_{hk} := D_{hk}^{\text{on}} \cup D_{hk}^{\text{off}}, \quad \text{for all } (h, k) \in \mathcal{E}^+, \tag{4a}$$

where

$$D_{hk}^{\text{on}} := \{a_{hk} = 0\} \cap \{2(y_h - y_k)^2 + 2R^2 \leq \tag{4b}$$
$$\frac{d_k + 1}{d_h} \sum_{\ell \neq k} \frac{a_{h\ell}}{d_\ell}(y_h - y_\ell)^2 + \frac{d_h + 1}{d_k} \sum_{\ell \neq h} \frac{a_{k\ell}}{d_\ell}(y_k - y_\ell)^2\}$$

$$D_{hk}^{\text{off}} := \{a_{hk} = 1\} \cap \{2(y_h - y_k)^2 - 2R^2 \geq \tag{4c}$$
$$\frac{d_k}{d_h - 1} \sum_{\ell \neq k} \frac{a_{h\ell}}{d_\ell}(y_h - y_\ell)^2 + \frac{d_h}{d_k - 1} \sum_{\ell \neq h} \frac{a_{k\ell}}{d_\ell}(y_k - y_\ell)^2\}.$$

Jump equations (3) should be understood in the sense that hybrid solutions only experience the change of one edge $(i, j) \in \mathcal{E}^+$ across one jump. This does not prevent multiple edges to be activated or deactivated at the same (ordinary) time, however such a simultaneous activation/deactivation is conveniently represented by multiple jumps of the hybrid solution. This elegant description enables studying the qualitative behavior of solutions by analyzing the change of a Lyapunov function across each single jump, namely across the change of only one edge a_{ij} under the condition that (y, a) belongs to D_{ij}.

In the selection of the jump sets D_{hk} of (4), $R > 0$ is a design parameter that allows tuning the sensitivity of the link activation and deactivation mechanisms. The rationale for this apparently involved definition is the following. A new connection between h and k is established when the distance $|y_h - y_k|$ is small compared to a weighted average of the distances between h (or k) and their current neighbors. On the contrary, a connection is dropped when the two individuals are too far apart, compared with their distance to their other neighbors. For this reason, the jump rule (4) does not prevent groups of individuals from being disconnected. Consequently, the solutions to dynamics (2)-(3) will not in general converge to a unique global consensus, but instead to disconnected clusters of individuals that share the same opinion.

The jump dynamics is finally conveniently written by compactly representing (3) by the update laws:

$$\begin{bmatrix} y^+ \\ a^+ \end{bmatrix} = g_{hk}(y, a),\ (y, a) \in D_{hk}, \quad \forall (h, k) \in \mathcal{E}^+, \tag{5}$$