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On solution of control problems for nonlinear systems on finite time interval ⋆

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Abstract: In this paper, a finite-dimensional nonlinear control system is considered on a finite time interval. Research deals with the problem of approach of the system with a target set in the system phase space at a fixed moment of time. A discrete approximation scheme is proposed for solution of the approach problem. Model examples of mechanical systems are given with application of the proposed scheme.

Keywords: optimal control, nonlinear systems, approach problem, resolvability sets, discrete approximation schemes

1. INTRODUCTION

A nonlinear and nonstationary system is considered in the Euclidean space R^n and on a finite time interval. We study the problem of approach of the system with a compact target set in R^n at the terminal moment within the considered time interval. The problem is split into two problems:
1. To find out is it possible to lead the system to the target set from a given initial position.
2. In the case when it is possible to bring the system to the target set from the given initial position, to construct an open-loop (programming) control providing this rapprochement.

In the paper, we discuss a method to construction of solutions for the approach problem which is based on usage of of the so-called resolvability set W (see Krasovskii and Subbotin (1974); Kurzhanskii (2009, 1999); Osipov (1971)). Construction of the resolvability set W in the space of positions (t, x) of the control system is implemented on the basis of attainability sets and integral funnels of control systems Kurzhanskii (2009); Kurzhanski and Valyi (1997); Chernousko (1988); Lee and Markus (1967). The set W is constructed approximately in the process of step-by-step backward calculations from the target set of the approach problem in the form of some finite set W^a in the space of positions of the control system. This set is used then in the procedure of construction of a resolving open-loop control. The procedure presumes usage of local (step-by-step) controls involved at the previous stage of solution of the approach problem in construction of the set W^a.

2. PROBLEM OF APPROACH

Let us consider on the finite time interval $[t_0, \vartheta]$, $t_0 < \vartheta < \infty$, the control system

$$\frac{dx}{dt} = f(t, x, u). \tag{1}$$

Here t – time variable, $x \in R^n$ – phase vector, $u \in P$ – vector of control, P – compact set in the Euclidean space R^p.

The system (1) is subject to the following conditions.

Condition 1.1. The function $f(t, x, u)$ is continuous on $[t_0, \vartheta] \times R^n \times R^p$, and for any bounded and closed domain $D \subset [t_0, \vartheta] \times R^n$ there exists such constant $L = L(D) \in (0, \infty)$ that

$$\|f(t, x^{(1)}, u) - f(t, x^{(2)}, u)\| \leqslant L\|x^{(1)} - x^{(2)}\|, \quad (t, x^{(i)}, u) \in D \times P, \quad i = 1, 2. \tag{2}$$

Here $\|f\|$ – norm of vector f in R^n.

Condition 1.2. There exists such constant $\gamma \in (0, \infty)$ that

$$\|f(t, x, u)\| \leqslant \gamma(1 + \|x\|), \quad (t, x, u) \in [t_0, \vartheta] \times R^n \times P. \tag{3}$$

Condition 1.3. The set $F(t, x) = f(t, x, P) = \{f(t, x, u) : u \in P\} \subset R^n$ is convex for all $(t, x) \in [t_0, \vartheta] \times R^n$.

Let us denote by symbol $X(t^*, t_*, x_*)$ $(x_* \in R^n, t_0 \leqslant t_* < t^* \leqslant \vartheta)$ the attainability set in R^n of system (1) which corresponds to moment t^* and initial condition $x(t_*) = x_*$. The symbol $X(t_*, x_*) = \bigcup\limits_{t^* \in [t_*, \vartheta])} (t^*, X(t^*, t_*, x_*))$ stands for the integral funnel of system (1) with initial position $(t_*, x_*) \in [t_0, \vartheta] \times R^n$. Here $(t^*, X^*) = \{(t^*, x^*) : x^* \in X^*\}$, $X^* \subset R^n$.

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Under conditions 1.1–1.3 the set $X(t^*, t_*, x_*)$ is the attainability set of the differential inclusion (d.i.)

$$\frac{dx}{dt} \in F(t,x), \quad x(t_*) = x_*. \tag{4}$$

The set $X(t^*, t_*, x_*)$ is a compact set in R^n for all $(t_*, x_*) \in [t_0, \vartheta] \times R^n$, $t^* \in [t_*, \vartheta]$, and the set $X(t_*, x_*)$ is compact in $[t_0, \vartheta] \times R^n$.

Assume that together with the system (1) a compact target set $M \subset R^n$ is given.

Problem 1. It is necessary to identify in the space $[t_0, \vartheta] \times R^n$ the set W of all those initial positions (t_*, x_*) of the system (1) for each of which there exists an admissible control $u(t)$, $t \in [t_0, \vartheta]$, leading the system (1) at the moment ϑ to the target set M (i.e. $x(\vartheta) \in M$).

Problem 2. Let $x^{(0)} \in R^n$. It is required to check does point $x^{(0)}$ satisfy the inclusion $(t_0, x^{(0)}) \in W$, and to construct an admissible control $u^*(t)$, $t \in [t_0, \vartheta]$, generating the motion $x^*(t)$, $x^*(t_0) = x^{(0)}$, of the system (1) for each $x^*(\vartheta) \in M$.

Since (precise) solutions of Problems 1 and 2 can be obtained in rare cases, then, in the paper, we focus on description of procedures for construction of approximate solutions of these problems [Ushakov et al. (2013)].

Let us describe a scheme of approximate solution for Problem 1.

By the symbol $Z(t_0, z^{(0)}) \subset [t_0, \vartheta] \times R^n$ we denote the integral funnel of the system

$$\frac{dz}{d\tau} = f^*(\tau, z, v), \quad z(t_0) = z^{(0)}, \quad \tau \in [t_0, \vartheta]. \tag{5}$$

Here $f^*(\tau, z, v) = -f(t_0 + \vartheta - \tau, z, v)$, $(\tau, z, v) \in [t_0, z, v] \times R^n \times P$.

The system (5) is the system (1) written in the so-called backward (reciprocal) time $\tau = t_0 + \vartheta - t$, $t \in [t_0, \vartheta]$. Let us denote by the symbol $Z = Z(t_0, M) = \bigcup_{z^{(0)} \in M} Z(t_0, z^{(0)})$ the integral funnel of the system (5) with the initial set $(t_0, M) = \{(t_0, z^{(0)}) : z^{(0)} \in M\}$.

Since sets W and Z in $[t_0, \vartheta] \times R^n$ are connected by the equality $W(t) = Z(\tau)$, $t = t_0 + \vartheta - \tau$, $\tau \in [t_0, \vartheta]$, then the set W can be calculated as the integral funnel $Z = Z(t_0, M)$ of the system (5).

Without loss of generality, we can assume that under conditions 1.1–1.3 the sets Z and W, and also approximate sets for them, emerging in our constructions, are contained in some bounded and closed cylindrical domain $D = [t_0, \vartheta] \times D^*$. Namely this domain D is considered in our constructions.

Also it should be mentioned that condition 1.2 is quite strong. Nevertheless it remains widespread in control theory. Condition 1.2 could be replaced with condition of uniformly bounded solutions $x(t)$ of system (1) on time interval $[t_0, \vartheta]$ with initial values $x(t_0)$ from arbitrary compact X_0 in R^n. This condition is less constructive and more general.

Let us introduce on the axis of the backward time τ a finite partition $\Gamma = \{\tau_0, \tau_1, \ldots, \tau_N = \vartheta\}$ of the time interval $[t_0, \vartheta]$ with steps $\Delta_i = \tau_{i+1} - \tau_i = \Delta > 0$, $i = 0, \ldots, N-1$.

In the ideal case, we want to calculate the integral funnel $Z = Z(t_0, M)$ as a finite series of cross-sections

$$Z(\tau_{i+1}) = Z(\tau_{i+1}, \tau_i, Z(\tau_i)), \quad i = 0, \ldots, N-1. \tag{6}$$

Here $Z(\tau^*, \tau_*, Z_*)$ is the attainability set of the system (5) at the time moment τ^* with the initial set (τ_*, Z_*), $Z_* \subset R^n$, corresponding to time moment $\tau_* \in [t_0, \tau^*]$.

Since, as a rule, we can not calculate precisely the sets $Z(\tau_{i+1})$ (6), we construct some their approximations $Z^a(\tau_{i+1})$ as finite sets in R^n.

Let us describe preliminary a construction scheme for some sub-series of finite sets $Z^a(\tau_{i+1})$ in R^n.

Namely, in accordance with Ushakov (2012) we determine the mapping $(\tau^*, \tau_*, Z_*) \mapsto Z^{(\delta)}(\tau^*, \tau_*, Z_*)$, Z_* is a finite set in R^n, $t_0 \leqslant \tau_* < \tau^* \leqslant \vartheta$, $\delta = \tau^* - \tau_*$, by the following relation

$$Z^{(\delta)}(\tau^*, \tau_*, Z_*) = \bigcup_{z_* \in Z_*} Z^{(\delta)}(\tau^*, \tau_*, z_*). \tag{7}$$

Here $Z^{(\delta)}(\tau^*, \tau_*, z_*) = z_* + \delta F^{(\delta)}(\tau_*, z_*)$, and the mapping $(\tau_*, z_*) \mapsto F^{(\delta)}(\tau_*, z_*)$ is some finite valued approximation of the mapping $(\tau_*, z_*) \mapsto F^*(\tau_*, z_*) = f^*(\tau_*, z_*, P)$ on the set D such that

$$\sup_{(\tau_*, z_*) \in D} d(F^{(\delta)}(\tau_*, z_*), F^*(\tau_*, z_*)) \leqslant \xi^*(\delta), \quad \delta \in (0, \infty). \tag{8}$$

Here function $\xi^*(\delta)$, $\delta \in (0, \infty)$, is chosen from the condition $\xi^*(\delta) \downarrow 0$ when $\delta \downarrow 0$.

Remark 1. The mapping $(\tau_*, z_*) \mapsto F^{(\delta)}(\tau_*, z_*)$ can be determined, particularly, by the equality $F^{(\delta)}(\tau_*, z_*) = f(\tau_*, z_*, P^{(\delta)})$, where $P^{(\delta)}$, $\delta \in (0, \infty)$, is some finite set in P satisfying relation (8).

Further, we assume that $Z^a(\tau_0)$ is some finite set in R^n such that

$$d(Z^a(\tau_0), Z(\tau_0)) = d(Z^a(\tau_0), M) \leqslant \sigma^*(\Delta).$$

Here function $\sigma^*(\delta)$, $\delta \in (0, \infty)$, satisfies the condition $\sigma^*(\delta) \downarrow 0$ when $\delta \downarrow 0$.

The sets $Z^a(\tau_{i+1})$, $i = 0, \ldots, N-1$, are determined in the recurrent procedure

$$Z^a(\tau_{i+1}) = Z^{(\Delta)}(\tau_{i+1}, \tau_i, Z^a(\tau_i)). \tag{9}$$

Remark 2. For the selected domain D the function

$$\begin{aligned} \omega^*(\delta) = \max\{&\|f(t_*, x, u) - f(t^*, x, u)\| : \\ &(t_*, x, u) \in D \times P,\ (t^*, x, u) \in D \times P, \\ &\|t_* - t^*\| \leqslant \delta\}, \quad \delta \in (0, \infty), \end{aligned}$$

satisfies relations $\omega^*(\delta) \downarrow 0$ when $\delta \downarrow 0$ and

$$\begin{aligned} &d(F(t_*, x_*), F(t^*, x^*)) \leqslant \omega^*(\delta) + L\|x_* - x^*\|, \\ &(t_*, x_*) \in D,\ (t^*, x^*) \in D,\ \|t_* - t^*\| \leqslant \delta. \end{aligned}$$

Here $L = L(D)$, and $d(F_*, F^*)$ is the Hausdorff distance between compact sets F_* and F^* in R^n.