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A Numerical Algorithm for Optimal Control of Systems with Parameter Uncertainty

Claire Walton * **Chris Phelps** ** **Qi Gong** *** **Isaac Kaminer** ****

* *Naval Postgraduate School, Monterey, CA, 93943 USA (e-mail: clwalton1 @ nps.edu).*
** *University of California Santa Cruz, Santa Cruz, CA 95060 USA (e-mail: cdphelps @ soe.ucsc.edu).*
*** *University of California Santa Cruz, Santa Cruz, CA 95060 USA (e-mail: qigong @ soe.ucsc.edu)*
**** *Naval Postgraduate School, Monterey, CA 93943 USA (e-mail: kaminer @ nps.edu)*

Abstract: This paper describes a numerical scheme for computing optimal solutions to a class of nonlinear optimal control problems in which parameter uncertainty may be a feature of the state dynamics or objective function. Consistency results are provided for states and controls generated by the algorithm as well as for the adjoint variables.

Keywords: optimal control, nonlinear control, numerical methods

1. INTRODUCTION

The issue of control problems given uncertainty has been studied extensively in recent decades. The two main challenges in such problems are stability and performance, and design necessities often force a compromise between the two. For engineering reasons, the guarantee of stability and minimization of worst-case performance has been an important focus of work in this area. However, though methods such as H_∞ optimization can guarantee stable answers for certain systems while also satisfying a specified performance criterion, prioritizing stability provides conservative solutions with respect to performance. These conservative methods may not in some cases actualize the full engineering possibilities of a system, especially when the uncertainties of control problem have known structural features such as static parameter uncertainty. As a contrasting approach, this paper supplies a numerical algorithm for optimizing the performance of a general class of nonlinear control problems containing parameter uncertainty. The class of problems addressed by this algorithm is defined as follows:

Problem P. Determine the function pair (x, u) with $x \in W_{1,\infty}([0,T] \times \Theta; \mathbb{R}^{n_x})$, $u \in L_\infty([0,T]; \mathbb{R}^{n_u})$ that minimizes the cost

$$J[x,u] = \int_\Theta \left[F\left(x(T,\theta),\theta\right) + \int_0^T r(x(t,\theta), u(t), t, \theta) dt \right] d\theta \tag{1}$$

subject to the dynamics

$$\frac{dx}{dt}(t,\theta) = f(x(t,\theta), u(t), \theta), \tag{2}$$

initial condition $x(0,\theta) = x_0(\theta)$, and the control constraint $g(u(t)) \le 0$ for all $t \in [0,T]$. The set $L_\infty([0,T]; \mathbb{R}^{n_u})$ is the set of all essentially bounded functions, $W_{1,\infty}([0,T] \times \Theta; \mathbb{R}^{n_x})$ the Sobolev space of all essentially bounded functions with essentially bounded distributional derivatives, and $F : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\theta} \mapsto \mathbb{R}$, $r : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R} \times \mathbb{R}^{n_\theta} \mapsto \mathbb{R}$, $g : \mathbb{R}^{n_u} \mapsto \mathbb{R}^{n_g}$. Additional conditions imposed on the state and control space and component functions are specified in Section 2.

In Problem **P**, the set Θ is the domain of a parameter $\theta \in \mathbb{R}^{n_\theta}$. The format of the cost functional is that of the integral over Θ of a Mayer-Bolza type cost with parameter θ, i.e. the integral over Θ of a cost of the form:

$$F\left(x(T,\theta),\theta\right) + \int_0^T r(x(t,\theta), u(t), t, \theta) dt.$$

This parameter can represent a range of values for a feature of the system, such as in ensemble control [Ruths and Li (2012)], or a stochastic parameter with a known probability density function. In the case of a stochastic parameter, this format, of the integration over Θ of a Mayer-Bolza cost, is one which can encompass expressions of quantities such as the expectation or variance of costs which depend on a parameter θ.

Instances of this class of problems have arisen in multiple recent applications. For example, the ensemble control problem [Li and Khaneja (2007); Becker (2012); Ruths and Li (2012)], deals with the control of a family or continuum of systems $x(t,\theta)$ whose dynamics depend continuously on a parameter θ. Optimizing the behavior of these systems over all parameter values creates control problems in the form of Problem **P**. Another area of recent interest is the optimal search problem [Walton et al. (2014); Phelps et al. (2012, 2014)], which aims to optimize the search for uncertain targets given a specified model for target uncertainty. When uncertain target motion is modeled as a deterministic function conditionally dependent on a set of unknown parameter values, as in [Foraker (2011)], the expected probability of success becomes dependent on the parameter set as well and a problem of this class is created. An additional example can be found in chemical

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engineering, where instances of Problem **P** have been utilized for the optimization of batch processes under uncertainty [Terwiesch et al. (1998); Ruppen et al. (1995)].

Variations of this problem were studied extensively in the 1970's, such as in [Lukka (1977)] and [Pursiheimo (1977)], with a focus on deriving analytical conditions for optimality. In [Gabasov and Kirillova (1974)] necessary conditions for Problem **P** (subject to the assumptions of Section 2) were established. These conditions, in the form of a Pontryagin-like minimum principle, are as follows:

Problem $\mathbf{P}^{\lambda}$. [Gabasov and Kirillova (1974), pp. 80-82]. If (x^*, u^*) is an optimal solution to Problem **P**, then there exists an absolutely continuous costate vector $\lambda^*(t,\theta)$ such that for $\theta \in \Theta$:

$$\frac{d\lambda^*}{dt}(t,\theta) = -\frac{\partial H(x^*, \lambda^*, u^*, t, \theta)}{\partial x}, \lambda^*(T,\theta) = \frac{\partial F(x^*(T,\theta),\theta)}{\partial x} \quad (3)$$

where H is defined as:

$$H(x,\lambda,u,t,\theta) = \lambda f(x(t,\theta),u(t),\theta) + r(x(t,\theta),u(t),t,\theta). \quad (4)$$

Furthermore, the optimal control u^* satisfies

$$u^*(t) = \underset{u \in U}{\arg\min} \mathbf{H}(x^*, \lambda^*, u, t),$$

where $\mathbf{H}$ is given by

$$\mathbf{H}(x,\lambda,u,t) = \int_{\Theta} H(x,\lambda,u,t,\theta)d\theta. \quad (5)$$

As can be seen, these conditions are of a complexity such that analytic solutions to problems are rare and restricted to simple cases. Thus, the focus in current years has shifted to supplying numerical solutions.

Recently, there has been much progress in developing numerical methods for tackling optimization problems with parameter uncertainties. For instance, for nonlinear finite-dimensional optimization problems, Robust Optimization (RO) frameworks have been developed to address the minimization of mean performance given constraints on variance or other risk metrics, such as in [Darlington et al. (2000)]. In optimal control, the method of polynomial chaos has been applied to a variety of problems with amenable problem structures, such as quadratic costs or linear dynamics, [Fisher and Bhattacharya (2011), Hover and Triantafyllou (2006)]. More general nonlinear control problems were approached by applying discretization-based methods in [Phelps et al. (2012, 2014)], where a consistent numerical method was provided for a class of problems with time-invariant parameter uncertainty effecting only the cost function; and in [Ruths and Li (2012)], where a multi-dimensional pseudospectral collocation scheme for discretizing both time and parameter space was developed for problems of the form of Problem **P**.

This paper will add to these options by extending the discretization-based methods of [Phelps et al. (2014)] to a larger class of problems. The problems tackled by the algorithm described in this paper are comparable in scope to those tackled in [Ruths and Li (2012)]. However, the algorithm in this paper provides several new benefits. The first is that it enables a much wider variety of numerical methods than just pseudospectral collocation to be applied as means of approximation, in both the parameter domains and time domain. Any discretization nodes with an associated convergent quadrature scheme [1] for integration can be used in the parameter domain. This allows, for example, for the computation of problems with high parameter space dimension, in which the full tensor product required by multi-dimensional pseudospectral collocation may be intractable. In these cases, alternate discretization methods may be chosen, for instance Smolyak sparse grid methods [Smolyak (1963)]. Additionally, multiple discretization methods are available for use in the time domain, with their suitability and convergence properties determined solely by their effectiveness on standard optimal control problems. Another benefit provided in this paper is a proof of the consistency of the dual problems created by discretization with respect to the dual problem of the original, i.e. Problem $\mathbf{P}^{\lambda}$. This consistency enables the use of the dual problems and their adjoint variables as verification tools and for theoretical analysis.

In the case of these discretization-based algorithms, there are three theoretical questions which need to be addressed: 1) Feasibility: Does the discretized problem create answers which satisfy the original problem constraints? 2) Convergence: Does the discretized problem converge as discretization is refined? 3) Consistency: If the discretized problem converges, does it in fact converge to the optimal solution of the original problem?

This paper addresses issues 1 and 3. Issue 2 is determined by the specifics of the chosen discretization schemes and is considered future work. The paper is organized as follows: Section 2 provides the main regularity assumptions used in conjunction with this numerical algorithm. Problem **P** as stated above is an informal statement and Section 2 elaborates on the additional conditions imposed on the component functions and state and control spaces. Section 3.1 formulates an approximation of Problem **P** which will be used as the foundation for the algorithm. Section 3.2 evaluates the consistency properties of the approximate problem as the number of nodes used in the approximation tends to infinity. Section 4 uses necessary conditions for both the original and approximate problems to discuss the consistency properties of the adjoint variables. Lastly, Section 5 demonstrates the proposed method by applying the numerical scheme to an uncertain optimal control problem.

2. REGULARITY ASSUMPTIONS

In order to address Problem **P**, we impose the following regularity assumptions:

Assumption 1. The function g is continuous and the set $U = \{\nu \in \mathbb{R}^{n_u} | g(\nu) \leq 0\}$ is compact.

In a real world scenario the set of allowable controls will be bounded and therefore U, being a closed and bounded set, will be compact.

Assumption 2. There exists a compact set $X \subset \mathbb{R}^{n_x}$ such that for each feasible [2] u and $\theta \in \Theta, t \in [0,T]$, $x(t,\theta) \in X$ where $x(t,\theta) = x_0 + \int_0^t f(x(s,\theta),u(s),\theta)ds$ for all $t \in [0,T]$.

[1] The specifics of the convergence properties required of the quadrature scheme are given in Assumption 4

[2] Feasible controls are those which satisfy all problem constraints, i.e. all $u \in L_\infty([0,T];\mathbb{R}^{n_u})$ such that $u(t) \in U$.