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Analysis of the Use of Low-Pass Filters with High-Gain Observers

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Abstract: High-gain observers play an important role in the design of feedback control for nonlinear systems. One challenge in the use of high-gain observers is the effect of measurement noise. It is shown in the literature that the presence of measurement noise puts a constraint on how high the observer gain could be, which forces a trade-off between the fast convergence of state estimates and the error due to measurement noise. A number of techniques have been proposed in the literature to attenuate the effects of measurement noise on systems utilizing a high-gain observer. In experimental applications of high-gain observers, it is quite common to use a low-pass filter to filter out the high-frequency components of the noise before feeding the measurement into the observer. In this paper we analyze effect of measurement noise on high-gain observers when low pass filters are used.

Keywords: High-gain observers, Nonlinear systems, Output feedback

1. INTRODUCTION

The use of high-gain observers is a well-known and effective method for state estimation and output feedback control of nonlinear systems. One problem associated with high-gain observers is the trade-off that exists between the fast convergence of state estimates and robustness to measurement noise. One idea to deal with this tradeoff is to use a higher gain during the transient period to achieve fast decay of the estimation error, and when the output estimation error is smaller than a certain threshold change the gain to a lower value to reduce the effect of measurement noise. This idea has been explored using gain switching, Ahrens and Khalil (2009), nonlinear gain, Prasov and Khalil (2013), and gain adaptation, Sanfelice and Praly (2011). In experimental applications of high-gain observers, it is quite common to use a Low-Pass Filter (LPF) to filter out the high-frequency components of the noise before feeding the measurement into the observer. There is no analysis in the literature when such filter is used. In this paper, we analyze the observer's estimation error when a low-pass filter is used. We also study the closed-loop system under output feedback control that includes a high-gain observer and a low pass filter.

2. OBSERVER PERFORMANCE IN THE PRESENCE OF MEASUREMENT NOISE

Consider a single-output nonlinear systems represented by

$$\dot{w} = f_0(w, x, u) \tag{1}$$

$$\dot{x}_i = x_{i+1}, \quad \text{for } 1 \le i \le \rho - 1 \tag{2}$$

$$\dot{x}_\rho = \phi(w, x, u) \tag{3}$$

$$y = x_1 + v \tag{4}$$

where $w \in R^\ell$ and $x = \text{col}(x_1, x_2, \dots, x_\rho) \in R^\rho$ form the state vector, $u \in R^m$ is the input, $y \in R$ is the measured output, and $v(t) \in R$ is the measurement noise. We assume that f_0 and ϕ are locally Lipschitz in their arguments, $u(t)$, and $v(t)$ are piecewise continuous functions of t, and $w(t)$, $x(t)$, $u(t)$, and $v(t)$ are bounded for all $t \ge 0$.[1] In particular, let $w(t) \in W \subset R^\ell$, $x(t) \in X \subset R^\rho$, $u(t) \in U \subset R^m$, and $|v(t)| \le N$ for all $t \ge 0$, for some compact sets W, X and U, and a positive constant N.

A high-gain observer that estimates x by $\hat{x}$ is given by

$$\dot{\hat{x}}_i = \hat{x}_{i+1} + \frac{\alpha_i}{\varepsilon^i}(y - \hat{x}_1), \quad \text{for } 1 \le i \le \rho - 1 \tag{5}$$

$$\dot{\hat{x}}_\rho = \phi_0(\hat{x}, u) + \frac{\alpha_\rho}{\varepsilon^\rho}(y - \hat{x}_1) \tag{6}$$

where ϕ_0 is a nominal model of ϕ, ε is a sufficiently small positive constant, and α_1 to α_ρ are chosen such that the roots of

$$s^\rho + \alpha_1 s^{\rho-1} + \cdots + \alpha_{\rho-1}s + \alpha_\rho = 0 \tag{7}$$

have negative real parts. We assume that ϕ_0 is locally Lipschitz in its arguments and globally bounded in $\hat{x}$; that is,

$$|\phi_0(\hat{x}, u)| \le M \tag{8}$$

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[1] The boundedness of w, x, and u is needed because the system is operated in open loop. It will not be required in Section 5 when the observer is used in feedback control.

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for all $\hat{x} \in R^\rho$ and $u \in U$.

Define the scaled estimation errors

$$\eta_i = \frac{x_i - \hat{x}_i}{\varepsilon^{\rho-i}}, \quad \text{for } 1 \le i \le \rho \tag{9}$$

It can be shown that $\eta = \text{col}(\eta_1, \eta_2, \ldots, \eta_\rho)$ satisfies the equation

$$\varepsilon\dot{\eta} = F\eta + \varepsilon B\delta(w, x, \hat{x}, u) - (1/\varepsilon^{\rho-1})Ev \tag{10}$$

where $B = \text{col}(0, \ldots, 0, 1)$,

$$F = \begin{bmatrix} -\alpha_1 & 1 & 0 & \cdots & 0 \\ -\alpha_2 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ -\alpha_{\rho-1} & & & 0 & 1 \\ -\alpha_\rho & 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad E = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{\rho-1} \\ \alpha_\rho \end{bmatrix}$$

and $\delta = \phi(w, x, u) - \phi_0(\hat{x}, u)$. The matrix F is Hurwitz by design because its characteristic equation is (7). In view of (8), there is a positive constant L, independent of ε, such that

$$|\delta(w, x, \hat{x}, u)| \le L \tag{11}$$

for all $w \in W$, $x \in X$, $\hat{x} \in R^\rho$, and $u \in U$. Let θ satisfy the equation

$$\varepsilon\dot{\theta} = F\theta - Ev \tag{12}$$

The estimation error $\tilde{x}_i$ is given by

$$\tilde{x}_i = \varepsilon^{\rho-i}\eta_i = \varepsilon^{\rho-i}\left(\eta_i - \frac{1}{\varepsilon^{\rho-1}}\theta_i\right) + \frac{1}{\varepsilon^{i-1}}\theta_i \tag{13}$$

In view of (11), it can been seen that the ultimate bound on $\|\eta - (1/\varepsilon^{\rho-1})\theta\|$ is $O(\varepsilon)$. Thus, the effect of measurement noise on the steady-state of the estimation error $\tilde{x}_i$ is captured by the term $(1/\varepsilon^{i-1})\theta_i$.

3. OBSERVER PERFORMANCE IN THE PRESENCE OF MEASUREMENT NOISE AND LPF

The output y is passed through a single-input–single-output low pass filter of the form

$$\tau\dot{z} = A_f z + B_f y, \quad y_f = C_f z \tag{14}$$

where $\tau << 1$ is the filter's time constant, $z \in R^r$, A_f is a Hurwitz matrix, and the filter's dc-gain is one; that is, $-C_f A_f^{-1} B_f = 1$. Feeding y_f into the high-gain observer, its equation is given by

$$\dot{\hat{x}}_i = \hat{x}_{i+1} + \frac{\alpha_i}{\varepsilon^i}(y_f - \hat{x}_1), \quad \text{for } 1 \le i \le \rho - 1 \tag{15}$$

$$\dot{\hat{x}}_\rho = \phi_0(\hat{x}, u) + \frac{\alpha_\rho}{\varepsilon^\rho}(y_f - \hat{x}_1) \tag{16}$$

Let q satisfy the equation

$$\tau\dot{q} = A_f q + B_f v, \quad v_f = C_f q \tag{17}$$

with $q(0) = 0$. Then, $p = z - q$ satisfies the equation

$$\tau\dot{p} = A_f p + B_f x_1 \tag{18}$$

with $p(0) = z(0)$. Let $\sigma_1 = p + A_f^{-1}B_f x_1$, $\sigma_i = \dot{\sigma}_{i-1}$, for $2 \le i \le \rho$, and $\xi = p^{(\rho)}$, where $p^{(j)}$ is the jth derivative of p. Then, σ_1 to σ_ρ and ξ satisfy the equations

$$\tau\dot{\sigma}_i = A_f\sigma_i + \tau A_f^{-1}B_f x_{i+1}, \quad \text{for } 1 \le i \le \rho - 1 \tag{19}$$

$$\tau\dot{\sigma}_\rho = A_f\sigma_\rho + \tau A_f^{-1}B_f\phi(w, x, u) \tag{20}$$

$$\tau\dot{\xi} = A_f\xi + B_f\phi(w, x, u) \tag{21}$$

By the boundedness of x and $\phi(w, x, u)$, it can be shown that ξ is ultimately bounded, uniformly in ε and τ, and σ_1 to σ_ρ are ultimately bounded by $O(\tau)$. With

$$\varphi_i = \frac{C_f\sigma_i + x_i - \hat{x}_i}{\varepsilon^{\rho-i}}, \quad \text{for } 1 \le i \le \rho \tag{22}$$

and using $-C_f A_f^{-1} B_f = 1$, it can be verified that $\varphi = \text{col}(\varphi_1, \ldots, \varphi_\rho)$ satisfies the equation

$$\varepsilon\dot{\varphi} = F\varphi + \varepsilon B[C_f\xi - \phi_0(\hat{x}, u)] - (1/\varepsilon^{\rho-1})Ev_f \tag{23}$$

Let ψ satisfy the equation

$$\varepsilon\dot{\psi} = F\psi - Ev_f \tag{24}$$

The estimation error $\tilde{x}_i$ is given by

$$\begin{aligned}\tilde{x}_i &= -C_f\sigma_i + \varepsilon^{\rho-i}\varphi_i \\ &= -C_f\sigma_i + \varepsilon^{\rho-i}\left(\varphi_i - \frac{1}{\varepsilon^{\rho-1}}\psi_i\right) + \frac{1}{\varepsilon^{i-1}}\psi_i \end{aligned} \tag{25}$$

By the ultimate boundedness of ξ and ϕ_0, it can be seen that the ultimate bound on $\|\varphi_i - (1/\varepsilon^{\rho-1})\psi_i\|$ is $O(\varepsilon)$. We have already seen that the ultimate bound on σ_i is $O(\tau)$. Thus, the effect of measurement noise on the steady-state of the estimation error $\tilde{x}_i$ is captured by the term $(1/\varepsilon^{i-1})\psi_i$.

To compare the effect of measurement noise on the steady-state estimation error with and without filter, we need to compare θ, which satisfies equation (12), with ψ, which satisfies equation (24). Equation (12) is driven by the measurement noise v while (24) is driven by the filtered noise v_f.

4. SIMULATION

Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 - 2x_2 + 0.25x_1^2 x_2 + 0.2\sin 2t, \\ y &= x_1 + v,\end{aligned}$$

It can be shown (see (Khalil, 2015, Example 11.1)) that the set $\Omega = \{1.5x_1^2 + x_1x_2 + 0.5x_2^2 \le \sqrt{2}\}$ is positively invariant. Therefore, for all $x(0) \in \Omega$, $x(t)$ is bounded. We use the high-gain observer

$$\begin{aligned}\dot{\hat{x}}_1 &= \hat{x}_2 + \frac{2}{\varepsilon}(y - \hat{x}_1), \\ \dot{\hat{x}}_2 &= \frac{1}{\varepsilon^2}(y - \hat{x}_1),\end{aligned}$$

where $\phi_0 = 0$. The low-pass filter is taken as

$$\frac{1}{\tau^2 s^2 + 2\tau s + 1}$$

The simulation is carried out with $\varepsilon = 0.01$ and initial conditions $x_1(0) = 1$, $x_2(0) = -1$, and $\hat{x}_1(0) = \hat{x}_2(0) = 0$. Three values of τ are used: 0.1, 0.01, and 0.001. The measurement noise v is generated using the Simulink "Uniform Random Number" block with limits ± 0.001 and sampling time 0.0008 seconds. Figure 1 compares the steady-state error $\tilde{x}_2$ with and without filter when $\tau = 0.01$. It shows the effectiveness of the filter in reducing the steady-state error. Figure 2 compares three different values of τ. It shows the transient and steady-state responses