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Non Lipschitz triangular canonical form for uniformly observable controlled systems

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Abstract: We study the problem of designing observers for controlled systems which are uniformly observable and differentially observable, but with an order larger than the system state dimension : we have only an injection, and not a diffeomorphism. We establish that they can be transformed into a triangular canonical form but with possibly non locally Lipschitz functions. Since the classical high gain observer is no longer sufficient, we review and propose other observers to deal with such systems, such as a cascade of homogeneous observers.

Keywords: uniform observability, differential observability, canonical observable form, finite-time observers, homogeneous observers, exact differentiators

1. INTRODUCTION

1.1 Context

A lot of attention has been dedicated to the construction of nonlinear observers. Although a general theory has been obtained for linear systems, very few general approaches exist for nonlinear systems. In particular, the theory of high gain (Khalil and Praly (2013) and references therein) and Luenberger (Andrieu and Praly (2013); Andrieu (2014)) observers have been developed for autonomous non linear systems but their extension to controlled systems is not straightforward.

For designing an observer for a system, a preliminary step is often required. It consists in finding a reversible coordinate transformation, allowing us to rewrite the system dynamics in a targeted form more favorable for writing and/or analyzing the observer. In presence of control, two tracks are possible depending on whether we consider the input as a simple time function, making the system time dependent or as a more involved (infinite dimensional) parameter, making the system a family of dynamical system, indexed by the control. Accordingly the transformation mentioned above is simply time-varying or input dependent. Moreover, along the later itself, with the input seen as a parameter, the strength of the input dependence of the transformation may vary.

For example, in (Hammouri and Morales (1990); Besançon et al. (1996)), the transformation can depend arbitrarily on the input with the objective of obtaining a targeted form which is state-affine up to input/output injection, or more generally as in Besançon (1999), a targeted form which has a triangular structure. The dependence may also be on the derivative of the inputs as proposed in Gauthier and Kupka (2001) with the so called phase-variable representation as targeted form.

Alternatively, we may impose the transformation not to depend on the input. This is the context of *uniformly observable systems*. For example Gauthier and Bornard (1981); Gauthier et al. (1992) propose this track to obtain, as targeted form, a so-called triangular canonical form for which a high-gain observer can be built. More precisely, as detailed below, it is known that this observer can be built when, together with the uniform (in the control) observability of the system (see (Gauthier and Kupka, 2001, Definition I.2.1.2) or Definition 2 below), the transformation, obtained from the strong differential observability (see (Gauthier and Kupka, 2001, Definition I.2.4.2) or Definition 1 below), is a diffeomorphism.

In this paper we study the case where we have uniform observability and strong differential observability, but the latter with an order larger than the system state dimension, implying that the transformation is at most an injective immersion, and not a diffeomorphism as above. We shall see that, in this case, the system dynamics can still be described by a triangular canonical form but with functions which may be non locally Lipschitz. This leads us to study observers able to cope with such an extreme context and, in particular, to propose a new observer made of a cascade of homogeneous observers.

1.2 Definitions

Consider a controlled system of the form :

$$\dot{x} = f(x) + g(x)u \quad , \qquad y = h(x) \tag{1}$$

where x is the state in $\mathbb{R}^n$, u is an input in $\mathbb{R}^p$ and y is a measured output in $\mathbb{R}$. Given an input time function $t \mapsto u(t)$, we denote $X(x,t)$ a solution of (1) going through x at time 0. We are interested in estimating $X(x,t)$ knowing y and u but only as long as $(X(x,t),u(t))$ is in a given compact set $\mathcal{C} \times U$. Let $\mathcal{S}$ be an open subset of $\mathbb{R}^n$ containing $\mathcal{C}$. We will use the following two notions of observability defined in Gauthier and Kupka (2001) :

Definition 1. (Differential observability). (See (Gauthier and Kupka, 2001, Definition I.2.4.2).) The system (1) is differentially observable of order m on $\mathcal{S}$ if the function :

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$$\mathbf{H}_m(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{m-1} h(x) \end{pmatrix}$$

is injective on $\mathcal{S}$. If it is also an immersion, the system is called strongly differentially observable. ■

Definition 2. (Uniform observability). (See (Gauthier and Kupka, 2001, Definition I.2.1.2).) The system (1) is uniformly observable on $\mathcal{S}$ if for any pair (x_a, x_b) in $\mathcal{S}^2$ with $x_a \neq x_b$, there is no C^1 function $u : [0, T) \to U$ such that

$$h(X(x_a, t)) = h(X(x_b, t))$$

for all $t \leq T$ such that $(X(x_a, s), X(x_b, s)) \in \mathcal{S}^2$ for all $s \leq t$. ■

In the case where $m = n$, i.e. $\mathbf{H}_n$ is a diffeomorphism, we have :

Proposition 1. (See Gauthier and Bornard (1981); Gauthier et al. (1992)) If the system (1) is uniformly observable and strongly differentially observable of order $m = n$, it can be transformed into the following triangular canonical form :

$$\begin{aligned} \dot{z}_1 &= z_2 + \mathfrak{g}_1(z_1)\, u \\ &\vdots \\ \dot{z}_i &= z_{i+1} + \mathfrak{g}_i(z_1, \ldots, z_i)\, u \\ &\vdots \\ \dot{z}_n &= \varphi_n(z) + \mathfrak{g}_n(z)\, u\ , \end{aligned} \tag{2}$$

where the functions $\mathfrak{g}_i$ are locally Lipschitz. ■

Such a triangular form named Lipschitz triangular form, with Lipschitz nonlinearities is fortunately the nominal case for the high gain paradigm.

But as we shall see in Section 2, when the system is strongly differentially observable of order $m > n$, triangularity is preserved but Lipschitzness is lost. Hence high gain observers as those presented in Gauthier et al. (1992) can no longer be used.

We thus present in Section 3 possible designs of observers for the triangular canonical form (2) with non-Lipschitz $\mathfrak{g}_i$. Everything is finally illustrated with an example in Section 4.

Notations.

(1) We define the signed power function as

$$\lfloor a \rceil^b = \mathtt{sign}(a)\, |a|^b$$

where b is a nonnegative real number. In the particular case where $b = 0$, $\lfloor a \rceil^0$ is actually any number in the set

$$\mathrm{S}(a) = \begin{cases} \{1\} & \text{if } a > 0 \\ [-1, 1] & \text{if } a = 0 \\ \{-1\} & \text{if } a < 0 \end{cases}$$

Namely, writing $c = \lfloor a \rceil^0$ means $c \in \mathrm{S}(a)$. Note that the set valued map $a \mapsto S(a)$ is upper semi-continuous with closed and convex values.

(2) For x in $\mathbb{R}^p$ with $p \geq i$, we denote

$$\mathbf{x}_i = (x_1, ..., x_i)\ .$$

and, for x in $\mathcal{S}$,

$$\mathbf{H}_i(x) = (h(x), \ldots, L_f^{i-1} h(x)) \tag{3}$$

2. IMMERSION CASE ($m > n$)

The specificity of the triangular canonical form (2) is not so much in its structure but more in the dependence of its functions $\mathfrak{g}_i$ and φ_n. Indeed, for any k, $\mathbf{H}_k(x)$ satisfies always :

$$\overset{\cdot}{\overbrace{\mathbf{H}_k(x)}} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & 1 \\ 0 & \ldots & \ldots & \ldots & 0 \end{pmatrix} \mathbf{H}_k(x) + \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ L_f^k h(x) \end{pmatrix} + L_g \mathbf{H}_k(x) u$$

To get (2), we need further the existence of a sufficiently smooth function φ_k satisfying

$$L_f^k h(x) \ = \ \varphi_k(\mathbf{H}_k(x)) \tag{4}$$

and of sufficiently smooth functions $\mathfrak{g}_i$ satisfying

$$L_g L_f^{i-1}(x) \ = \ \mathfrak{g}_i(h(x), \ldots, L_f^{i-1} h(x)) \tag{5}$$

and this at least for all x in $\mathcal{C}$ if not in $\mathcal{S}$.

Let us illustrate via the following elementary example what can occur.

Example 1. Consider the system

$$\dot{x}_1 \ = \ x_2\ , \quad \dot{x}_2 \ = \ x_3^3\ , \quad \dot{x}_3 \ = \ 1 + u\ , \quad y \ = \ x_1$$

It is uniformly observable, differentially observable of order 3 and strongly differentially observable of order 5 since

$$\begin{aligned} \mathbf{H}_3(x) &= (h(x), L_f h(x), L_f^2 h(x)) \ = \ (x_1, x_2, x_3^3) \\ \mathbf{H}_5(x) &= (\mathbf{H}_3(x), L_f^3 h(x), L_f^4 h(x)) \ = \ (\mathbf{H}_3(x), 3x_3^2, 6x_3) \end{aligned}$$

where $\mathbf{H}_3$ is a bijection and $\mathbf{H}_5$ is an injective immersion on $\mathbb{R}^3$. For $\mathbf{H}_3$, we have

$$L_f^3 h(x) \ = \ 3x_3^2 \ = \ 3(L_f^2 h(x))^{2/3}$$

Hence there is no Lipschitz function φ_3 satisfying (4). Similarly, for $\mathbf{H}_5$, we have

$$L_g L_f^2 h(x) = 3x_3^2 = L_f^3 h(x) = 3(L_f^2 h(x))^{2/3}$$

so there is no locally Lipschitz function $\mathfrak{g}_3$ satisfying (5).■

Concerning the existence of continuous functions φ_k and $\mathfrak{g}_i$ we have the following results given without proof due to space limitations.

Proposition 2. Suppose the system (1) is differentially observable of order m on an open set $\mathcal{S}$ containing the given compact set $\mathcal{C}$. There exists a continuous function $\varphi_m : \mathbb{R}^m \to \mathbb{R}$ satisfying (4) for all x in $\mathcal{C}$. If the system (1) is strongly differentially observable of order m on $\mathcal{S}$, the function φ_m can be chosen Lipschitz on $\mathbb{R}^m$. ■

Proposition 3. Suppose the system (1) is uniformly observable on an open set $\mathcal{S}$ containing the given compact set $\mathcal{C}$, then, for all i, there exist continuous functions $\mathfrak{g}_i : \mathbb{R}^i \to \mathbb{R}$ satisfying (5) for all x in $\mathcal{C}$. ■

Note that the values of φ_m and $\mathfrak{g}_i$ are only imposed on the compact set $\mathbf{H}_m(\mathcal{C})$. In particular, their behavior when $|z|$ tends to infinity is free and can be chosen to satisfy some extra constraints given by the observer design (see Assumption (7) in Section 3).

Note also that no assumption on $\mathbf{H}_m$ is needed for Proposition 3 to hold. But it says nothing on the regularity of the