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The Hamiltonian Formalism for Problems of Group Control Under Obstacles *

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Abstract: This paper suggests a unified method of solving target team control problems using methods of the Hamiltonian formalism. Considered is the problem of steering a team of m members towards a given target set while avoiding external obstacles. The team members must avoid collisions of their respective safety zones while remaining close to each other within a virtual ellipsoidal container or a chain of such containers. The container trajectory is therefore constructed firstly, then used as an external state constraint for the team. Solvability conditions for such target team control problem are formulated with routes for calculating respective control strategies indicated.

Keywords: Coordinated cooperative control, closed-loop control strategies, non-linear feedback, Hamiltonian formalism

1. INTRODUCTION

The problems of designing feedback strategies for teams of controlled motions are an important part of modern mathematical control theory. These problems are motivated by numerous examples from applied areas. The problem considered in this paper deals with steering a team of m members towards a given target set while avoiding collisions with each other and with external obstacles. Such problems arise when teams of autonomous vehicles are employed for solving tasks such as extinguishing fires (especially in mountainous terrain), finding underwater ship wreckage or defects in pipelines (see, for example, Somasundaram and Baras (2009) or Pettersen et al. (2006)). Another example is the problem of surveying the asteroid belt of our solar system by teams of small autonomous spacecrafts. There is also an extensive research on applying control theory to collective behaviour of biological systems like herds of animals, flocks of birds or schools of fish (see Olfati-Saber (2006)).

The basic problem considered here is for a team of non-colliding motions to jointly reach a given target while avoiding pre-specified external obstacles. To meet this requirement each team member is described by a ball with its own *safety zone* where no other team member would be present. On the other hand it is desirable to keep all the team members near each other for communication reasons, so that they would be within sensor range of one another. To cope with such constraints, we require the team to move within a virtual container described by a tube (a set-valued map with compact convex values) or a chain of such containers of smaller size when the gap between the obstacles is small. The joint motion is of course the fastest when one container is sufficient.

In this paper a rigorous approach to solving such a problem is presented. Firstly, we state the main mathematical problem — the one of *target team control*: to find a virtual tube for the container and a feedback control strategy for each team member that steer the team to the target set while avoiding external obstacles and ensuring the absence of internal collisions within the team. We then separate this problem into two sub-problems: the one of describing the ellipsoidal container tube responsible for avoiding external obstacles and the other of obtaining the feedback controls for team members under collision avoidance and joint external state constraint enforced now by the container tube. The later problem is then separated once more into three stages based on the passage between obstacles.

It is important to emphasize that the described new types of feedback control problems are formulated on finite time-intervals and their solutions are new mathematical challenges that should also include design of appropriate numerical techniques.

The present work is based on results obtained in papers by Kurzhanski (2015) and Kurzhanski and Varaiya (2010). The discussed methods rely on Hamiltonian techniques for systems with matrix-valued state variable. Here such matrix may either describe the configuration of an ellipsoid or be constructed of state vectors for the team. Solvability conditions for the target team control problem are obtained in the form of a chain of Hamilton–Jacobi–Bellman equations. Control solution feedback strategies are then proposed for the team.

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Obtained relations thus indicate how to solve considered problems to the end. This is done by applying the theory of trajectory tubes (see Kurzhanski and Varaiya (2014)).

2. THE SYSTEM

On a finite time interval $[t_0, \theta]$ consider a team of m members with following Newtonian dynamics:

$$\ddot{x} = f(t, x, \dot{x}, u),\ t \in [t_0, \theta].$$

Here $x \in \mathbb{R}^{n\times m}$ is the matrix state composed of m-dimensional state vectors for the team,

$$x = [x^{(1)}, x^{(2)}, \ldots, x^{(m)}],\ x^{(j)} \in \mathbb{R}^n,$$

and $u \in \mathbb{R}^{p\times m}$ is the control matrix,

$$u = [u^{(1)}, u^{(2)}, \ldots, u^{(m)}],\ u^{(j)} \in \mathcal{P} \subset \mathbb{R}^p,$$

with $\mathcal{P}$ being a symmetric convex compact set ($\mathcal{P} = -\mathcal{P}$). Denote

$$\mathbf{x}^{(j)} = [x^{(j)\prime}, \dot{x}^{(j)\prime}]' \in \mathbb{R}^{2n},\ \mathbf{X} = [\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(m)}] \in \mathbb{R}^{2n\times m}.$$

A team member $x^{(j)}$ is described as a ball of radius r (a *safety zone*) around $x^{(j)}$, $\mathcal{B}_r\left(x^{(j)}\right) = x^{(j)} + \mathcal{B}_r(0)$, where

$$\mathcal{B}_r(c) = \left\{x :\ \langle x - c, x - c\rangle \leqslant r^2\right\}.$$

To ensure the safety of team members their safety zones must not overlap, we hence introduce the distance matrix $\mathbf{D}_r[t] = \{D_{ij}[t]\}_{i,j=1}^m$,

$$D_{ij}[t] = D\left(\mathcal{B}_r\left(x^{(i)}(t)\right), \mathcal{B}_r\left(x^{(j)}(t)\right)\right),$$

where $D(X, Y)$ is the Euclidian distance between two convex compact sets:

$$D(X, Y) = \min\{\|x - y\| \mid x \in X,\ y \in Y\}.$$

We thus have $D_{ij}[t] = \max\{0, \left\|x^{(i)}(t) - x^{(j)}(t)\right\| - 2r\}$.

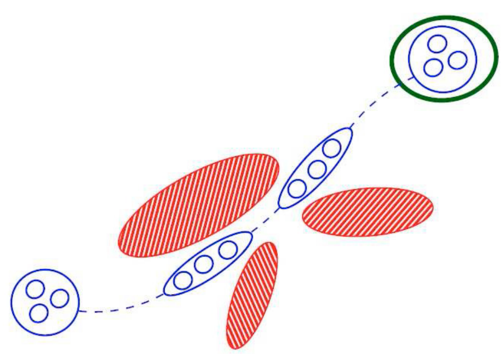

Fig. 1. The team within the container

Now the problem of target team control may be formulated as follows.

The problem of target team control (TTC). Design a closed-loop control strategy u and a set-valued map

$$\mathbf{E}[t] :\ [t_0, \theta] \to \operatorname{conv}\mathbb{R}^n$$

(a *virtual container*) such that the following conditions are fulfilled:

(1) The team is steered towards the target set $\mathcal{M}$:
$$\mathbf{X}[t_0] \xrightarrow[u(\cdot)]{} \mathbf{X}[\theta] \subseteq \mathcal{M} + \varepsilon\mathcal{B}_1(0).$$
(2) Collisions between team members are avoided in view of safety radius r:
$$D_{ij}[t] \geqslant 0,\ i \neq j,$$
(3) The *team proximity condition* holds:
$$\mathcal{B}_r\left(x^{(j)}(t)\right) \subset \mathbf{E}[t],\ j = 1, \ldots, m,\ t \in [t_0, \theta].$$
(4) Collisions with external obstacles $\mathbf{E}_k$ are avoided:
$$D(\mathbf{E}[t], \mathbf{E}_k) > 0,\ k = 1, \ldots, k_0,\ t \in [t_0, \theta].$$

Here the container $\mathbf{E}[t]$ can be chosen as a ellipsoidal-valued function $\mathcal{E}(q(t), Q(t))$ (an *ellipsoidal tube*),

$$\mathcal{E}(q(t), Q(t)) = \left\{x :\ \left\langle x - q(t), Q^{-1}(t)(x - q(t))\right\rangle \leqslant 1\right\},$$

where $q(t) \in \mathbb{R}^n$, $Q(t) \in \mathbb{R}^{n\times n}$, $Q(t) = Q'(t) > 0$.

In order to fit between the obstacles the container may have to be separated into several smaller containers, or even into a “chain” of containers — one for each team member. The fastest motion in time is when there is only one container. In the last case the team must follow the containers' center $q(t)$ as a chain.

The containers' volume must be kept within given limits to have space for placing the team:

$$\lambda_- \leqslant \operatorname{Vol}\mathbf{E}[t] \leqslant \lambda_+,\ t \in [t_0, \theta],\ 0 < \lambda_- < \lambda_+.$$

We presume that obstacles $\mathbf{E}_k$ are ellipsoids $\mathcal{E}(q_k, Q_k)$, $k = 1, \ldots, k_0$.

The suggested solution approach to the TTC problem consists of two following steps:

(1) First construct the tube $\mathbf{E}[t]$ which avoids the obstacles while keeping the necessary volume (see Section 3);
(2) Treating $\mathbf{E}[t]$ as a joint external phase constraint for the team, design the controls for each team member (see Section 4). These controls will be constructed in three stages in order to cope with the obstacles.

3. THE MOTION OF THE CONTAINER

Consider an ellipsoidal container $\mathbf{E}[t] = \mathcal{E}(q(t), Q(t))$ with the following dynamics:

$$\ddot{q}(t) = v,\ t \in [t_0, \theta],\ q(t_0) = q_0,\ \dot{q}(t_0) = v_0,$$
$$\dot{Q}(t) = T(t)Q + QT'(t) + B(t)V(t)B'(t),\ Q(t_0) = Q_0.$$

Here v and V are controls bounded by constraints

$$\langle v, v\rangle \leqslant \mu^2,\ \langle V, V\rangle \leqslant \mu^2.$$

and the matrix product is treated as $\langle A, B\rangle = \operatorname{tr} AB'$. Also assume that the target set is an ellipsoid as well, $\mathcal{M} = \mathcal{E}(m, M)$. We further presume that the system is completely controllable (for controllability conditions of matrix equations, see Chebunin (2003)).

The main control problem for the container can be formulated as follows: find feedback control strategies v, V that steer the container towards the target set,

$$\mathbf{E}[t] \subseteq \mathcal{E}\left(m, M(1 + \varepsilon^2)\right)$$

under minimal $\varepsilon > 0$, while avoiding collisions with the external obstacles:

$$D(\mathbf{E}[t], \mathbf{E}_k) > 0,\ k = 1, \ldots, k_0.$$

Recall that a *solvability set* for an arbitrary controlled system with state constraints represented by a set-valued function $\mathcal{Y}(\cdot)$ is defined as