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Analysis of semi-implicit DGFEM for nonlinear convection–diffusion problems on nonconforming meshes ☆

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Dedicated to Professor Ivo Babuška on the occasion of his 80th birthday.

Abstract

The paper deals with the numerical analysis of a scalar nonstationary nonlinear convection–diffusion equation. The space discretization is carried out by the discontinuous Galerkin finite element method (DGFEM), on general nonconforming meshes formed by possibly nonconvex elements, with nonsymmetric treatment of stabilization terms and interior and boundary penalty. The time discretization is carried out by a semi-implicit Euler scheme, in which the diffusion and stabilization terms are treated implicitly, whereas the nonlinear convective terms are treated explicitly. We derive a priori asymptotic error estimates in the discrete $L^\infty(L^2)$-norm, $L^2(H^1)$-seminorm and $L^\infty(H^1)$-seminorm with respect to the mesh size h and time step τ. Numerical examples demonstrate the accuracy of the method and manifest the effect of nonconvexity of elements and nonconformity of the mesh.

Keywords: Nonlinear convection–diffusion equation; Discontinuous Galerkin finite element method; Nonsymmetric treatment of stabilization terms – NIPG method; Interior and boundary penalty; Semi-implicit scheme; A priori error estimates; Experimental order of convergence

1. Introduction

The numerical solution of nonlinear conservation laws, convection–diffusion problems and flow problems requires the application of efficient, robust and accurate methods allowing to overcome various difficulties, as the precise capturing and resolution of boundary layers, shock waves and contact discontinuities. It is possible to say that nowadays in computational fluid dynamics (CFD) two techniques compete: the finite volume (FV) schemes and stabilized finite element methods (FEM). A survey of FV as well as FE approaches to the numerical simulation of compressible flow can be found, e.g. in [23].

A natural generalization of the FV and FE techniques is the *discontinuous Galerkin finite element method* (DGFEM), which appears to be very suitable for problems with solutions containing discontinuities and/or steep gradients. The DGFEM is based on piecewise polynomial but discontinuous approximations. It uses advantages of the FV as well as FE methods. Similarly as in the finite volume method, the DGFEM uses discontinuous approximations and boundary fluxes are evaluated with the aid of a numerical flux, which allows a precise capturing of discontinuities and steep gradients. Similarly as in the finite element method, the DGFEM uses higher degree polynomial approximations of solutions, which produces an accurate resolution in regions, where the solution is smooth.

There are several variants of the DGFEM for the solution of problems containing diffusion terms. It is possible to use primitive variables or a mixed method. The

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method can be stabilized with the aid of a symmetric or nonsymmetric treatment of diffusion terms, often combined with an interior and boundary penalty. We consider here the nonsymmetric variant with the interior and boundary penalty (denoted as NIPG method). This stabilization technique was proposed in [3,5] and represents the generalization of the boundary penalty by Babuška and Zlámal allowing to impose the Dirichlet boundary condition in a weak sense instead of building it in the finite element space (see [2]). The nonsymmetric variant was also investigated in [10,8,9,29] for elliptic and parabolic problems and in [19,20] for nonlinear convection–diffusion problems. Although this approach does not give an optimal order of convergence for elliptic problems, it leads to a coercive operator for an arbitrary positive penalty coefficient. This property is important when the DGFEM is applied to the system of the Navier–Stokes equations, where the numerical analysis is rather complicated, see [16].

There is a number of works devoted to theory and applications of the DGFEM. Let us mention, e.g. [1,3,5, 4,7,14–16,19,23–25,27,29,31]. For a survey of various discontinuous Galerkin techniques, see, e.g. [12,13].

In [17,20] we carried out a discretization of a scalar nonstationary convection–diffusion equation with nonlinear convective terms by the DGFEM with respect to space variables (the method of lines) and derived a priori error estimates. The time discretization can be carried out by the (explicit) Runge–Kutta methods, which are simple for implementation, but the resulting schemes are conditionally stable and the time step is drastically limited by the CFL stability condition. In order to avoid this disadvantage, it seems suitable to apply an implicit method, which allows us to use a much longer time step. However, a fully implicit DGFEM leads to a large, strongly nonlinear algebraic system, whose solution is rather complicated. This is the reason that in the present paper, which is a continuation of [20], we propose a semi-implicit scheme, which appears quite efficient and robust. The linear diffusion and stabilization terms are treated implicitly, whereas the nonlinear convective terms explicitly. Similarly as in [20] we allow to use a nonconforming mesh formed by nonconvex star-shaped polyhedral elements. In this paper we shall be concerned with theoretical analysis of error estimates of the semi-implicit method and present several numerical experiments verifying the theoretical results. Also the effect of nonconvexity of elements and nonconformity of a mesh will be treated in numerical experiments.

The contents of the paper is the following. In Section 2, the initial–boundary value problem for a scalar nonlinear convection–diffusion equation is formulated. In Section 3, we carry out the discretization of the problem by the semi-implicit DGFEM and establish the existence and uniqueness of the numerical solution. Section 4 contains some auxiliary results, namely assumptions on the space discretization (allowing even nonconforming grids with nonconvex star-shaped elements) and some important inequalities and estimates. These results are used in Section 5, where error estimates in the discrete $L^\infty(L^2)$-norm, $L^2(H^1)$-seminorm and $L^\infty(H^1)$-seminorm are proven. We obtain also estimates of the error in the penalty terms. In Section 6 we present numerical examples demonstrating the accuracy and robustness of the DGFEM. In Section 7 we introduce some concluding remarks and formulate open problems.

2. Continuous problem

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded polyhedral domain and $T > 0$. (For $d = 2$ under the concept of a polyhedral domain we mean a polygonal domain.) We set $Q_T = \Omega \times (0, T)$. By $\overline{\Omega}$ and $\partial\Omega$ we denote the closure and boundary of Ω, respectively. Let us consider the following *initial–boundary value problem*: Find $u : Q_T \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} = \varepsilon\Delta u + g \quad \text{in } Q_T, \tag{1}$$

$$u|_{\partial\Omega\times(0,T)} = u_D, \tag{2}$$

$$u(x,0) = u^0(x), \quad x \in \Omega. \tag{3}$$

We assume that the data satisfy the following conditions:

$$\begin{aligned}
&\text{(a)} \quad f_s \in C^1(\mathbb{R}), \quad f_s(0) = 0, \quad s = 1, \ldots, d,\\
&\text{(b)} \quad \varepsilon > 0,\\
&\text{(c)} \quad g \in C([0,T]; L^2(\Omega)),\\
&\text{(d)} \quad u_D \text{ is the trace of some}\\
&\qquad\quad u^* \in C([0,T]; H^1(\Omega)) \cap L^\infty(Q_T)\\
&\qquad\quad \text{on } \partial\Omega \times (0,T),\\
&\text{(e)} \quad u^0 \in L^2(\Omega).
\end{aligned} \tag{4}$$

We use the standard notation for function spaces (see, e.g. [26]): $L^p(\Omega)$, $L^p(Q_T)$ denote the Lebesgue spaces, $W^{k,p}(\Omega)$, $H^k(\Omega) = W^{k,2}(\Omega)$ are the Sobolev spaces, $L^p(0,T;X)$ is the Bochner space of functions p-integrable over the interval $(0,T)$ with values in a Banach space X, $C([0,T];X)$ ($C^1([0,T];X)$) is the space of continuous (continuously differentiable) mappings of the interval $[0,T]$ into X.

The assumption that $f_s(0) = 0$, $s = 1, \ldots, d$, does not cause any loss of generality, as can be seen from Eq. (1). The functions f_s, called fluxes, represent convective terms, $\varepsilon > 0$ is the diffusion coefficient.

We shall assume that problem (1)–(3) has a weak solution (cf. [20], Section 2), satisfying the regularity conditions

$$\begin{aligned}
&u \in L^\infty(0,T; H^{p+1}(\Omega)),\\
&\frac{\partial u}{\partial t} \in L^\infty(0,T; H^{p+1}(\Omega)),\\
&\frac{\partial^2 u}{\partial t^2} \in L^\infty(0,T; L^2(\Omega)),
\end{aligned} \tag{5}$$

where an integer $p \geqslant 1$ will denote a given degree of polynomial approximations. Such a solution satisfies problem (1)–(3) pointwise. Under (5),

$$u \in C([0,T]; H^{p+1}(\Omega)), \quad \frac{\partial u}{\partial t} \in C([0,T]; L^2(\Omega)). \tag{6}$$