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A multi-level stabilized finite element method for the stationary Navier–Stokes equations ☆

Jian Li [a,b], Yinnian He [a,*], Hui Xu [c]

[a] Faculty of Science, Xi'an Jiaotong University, Xi'an 710049, PR China
[b] Department of Mathematics, Baoji University of Arts and Sciences, Baoji 721007, PR China
[c] School of Energy and Power Engineering, Xi'an Jiaotong University, Xi'an 710049, PR China

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Abstract

This paper proposes and analyzes a multi-level stabilized finite element method for the two-dimensional stationary Navier–Stokes equations approximated by the lowest equal-order finite element pairs. The method combines the new stabilized finite element method with the multi-level discretization under the assumption of the uniqueness condition. The multi-level stabilized finite element method consists of solving the nonlinear problem on the coarsest mesh and then performs one Newton correction step on each subsequent mesh thus only solving one large linear systems. The error analysis shows that the multi-level stabilized finite element method provides an approximate solution with the convergence rate of the same order as the usual stabilized finite element solution solving the stationary Navier–Stokes equations on a fine mesh for an appropriate choice of mesh widths: $h_j \sim h_{j-1}^2, j = 1, \ldots, J$. Moreover, the numerical illustrations agree completely with the theoretical expectations. Therefore, the multi-level stabilized finite element method is more efficient than the standard one-level stabilized finite element method.

AMS(MOS) subject classification: 35Q10; 65N30; 76D05

Keywords: Navier–Stokes equations; Stabilized finite element method; Inf–sup condition; Multi-level discretization

1. Introduction

The development of stable mixed finite element methods is a fundamental component in the search for the efficient numerical methods for solving the Navier–Stokes equations governing the flow of an incompressible fluid by using a primitive variable formulation. The importance of ensuring the compatibility of the component approximations of velocity and pressure by satisfying the so-called inf–sup condition is widely understood. The numerous mixed finite elements satisfying the inf–sup condition have been proposed over the years. However, elements not satisfying the inf–sup condition may also work well. Some kinds of mixed finite elements which violate the inf–sup condition, are very attractive and usable on many occasions. In particular, the equal-order mixed finite elements are of practical importance in scientific computation because it is computationally convenient in a parallel processing and multigrid context. To the author's knowledge, some numerical results show that better results can be achieved by the lowest equal-order finite element pairs than other unstable lowest order finite element pairs by using the given stabilized method for the stokes equation in [13,19]. Therefore, more attention has been attracted by the equal-order finite elements.

Admittedly, the most convenient choice of the finite element space from an implementational point of view would

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* Corresponding author.
E-mail addresses: jiaaanli@gmail.com (J. Li), heyn@mail.xjtu.edu.cn (Y. He).

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be the elements of the equal polynomial order in the velocity and the pressure with an identical degree distribution for both the velocity and the pressure. However, on that condition the inf–sup condition does not satisfy. Furthermore, the violation of the inf–sup condition often leads to nonphysical pressure oscillations.

In order to make fully use of these equal-order finite elements which do not satisfy the inf–sup condition, a popular strategy is to use the stabilized techniques to circumvent or ameliorate the compatibility condition. Some kinds of methods have been studied during the past decades for the equal-order finite elements (see [1–7]). These methods contribute the stabilization for the Stokes equation by using the equal-order finite element pairs. The main drawback, however, is that the stabilization necessarily introduces the stabilized parameters either explicitly or implicitly. Also, some methods are conditionally stable and achieve suboptimal accuracy depending on the stabilized parameters with respect to the solution regularity (see [4,15]). Thus, insensitivity to such parameter values is important if the methodology is to be competitive.

The idea of the new stabilized finite element method based on two local Gauss integrations technique is derived from the work of [12,19,28,29] for the stationary Stokes equations. This stabilized method belongs to the local pressure projection method. We mostly borrow from the technique of [12] in theory. Unlike the penalty methods (see [8–11]) which uncouple pressure and velocity, stabilization aims at relaxing the continuity equation so as to apply the inf–sup condition to the incompatible spaces. Consistently, the stabilized methods (see [2,4,14]) are accomplished by using the residual of the momentum equation. These residual terms must be designed by the mesh-dependent parameters, whose optimal values are often not known. Especially to the lowest equal-order pairs, pressure and velocity derivatives in this residual either vanish or are poorly approximated, causing difficulties in the application of consistent stabilization. Another method is non-residual stabilization. Examples include local and global pressure jump formulations where the constraint is relaxed by using the jumps of the pressure across element interfaces. This stabilization requires the edge-based data structures, and the subdivision of the mesh into patches for the local jump formulation. However, the new method based on two local Gauss integrations technique do not require approximation of derivatives, specification of mesh-dependent parameters, and always lead to symmetric problems. Also, this stabilization is completely local at the element level. In addition, no edge-based data structures and assembly are required. As a result, the new stabilized method can be deployed in existing codes with very little additional effort by using simple Gauss integral technique. At the same time, we offer numerical results to compare the new method with other methods in [12,30,27]. The date indicate that the new method is more simple and efficient than others.

However, when the nonlinear Navier–Stokes equations are numerically solved, it takes much time. A common choice for this problem is two-level method or multi-level method. The basic idea of these methods is to compute an initial approximation on a very coarse mesh (involving the solution of a very small number of nonlinear equations). Moreover, the fine structures are captured by solving one linear system (linearized about the coarse mesh approximation using Newton iterative method) on a fine mesh. Some classical method of two-level and multi-level can be found in the works of Xu [20,21], Layton and Tobiska [18], Layton [22], Layton and Lenferink [23], Layton, Lee and Peterson [24], He [25,26,35] and Li [30,31]. The main idea of multi-level finite element method consists of solving the fully nonlinear Navier–Stokes problem only on the coarsest mesh; subsequent approximations are formed by solving the linearized Navier–Stokes problem about the solution on the previous level. Hence, the multi-level finite element method can save the large amount of computational time than the one-level finite element method.

As noted in the works cited above, the efficient method for the stationary Navier–Stokes equations by using the equal-order finite element pairs, is to combine the new stabilized finite element method with the multi-level discretization under the assumption of the uniqueness condition.

This paper firstly recalls the stabilized finite element method based on two local Gauss integrations technique [12] for the stationary Navier–Stokes equations approximated by the lowest equal-order finite elements $P_1 - P_1$ or $Q_1 - Q_1$. Then, we present the well-posedness and the optimal error estimate of the stabilized finite element method for the stationary Navier–Stokes equations in [30]. Finally, the results of Theorems 4.2 and 4.3 show that the method we study are of the convergence rate of the same order as the usual stabilized finite element method. However, our method is more efficient than the one-level finite element method.

The remainder of the paper is organized as follows. In the next section, abstract functional setting of the Navier–Stokes problem is given with some basic statements. The stabilized finite element approximations are recalled in Section 3. Error estimates on the multi-level method for the stabilized finite element solution (u_h, p_h) are derived from Section 4. In Section 5, a series of numerical experiments confirm the theoretical results completely.

2. Function setting of the Stationary Navier–Stokes problem

Let Ω be a bounded domain in R^2, assumed to have a Lipschitz-continuous boundary Γ and to satisfy a further condition stated in $(A1)$ below. The stationary Navier–Stokes equations are considered as follows:

$$\nu\Delta u + \nabla p + (u\cdot\nabla)u + \frac{1}{2}(\mathrm{div}u)u = f, \quad \mathrm{div}u = 0 \text{ in } \Omega, \tag{2.1}$$

$$u|_{\partial\Omega} = 0, \tag{2.2}$$