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Slack Variables Generation via QR Decomposition for Adaptive Nonlinear Control of Affine Underactuated Systems

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Abstract: This paper presents a slack variable generation method utilizing QR decomposition for an adaptive nonlinear controller of affine underactuated systems. Slack variables are adopted to overcome nonsquare properties of underactuated systems. QR decomposition has an advantage of fast and accurate calculation to compute least square solution of underdetermined systems. In this paper, the slack variable generation using the QR decomposition is proposed to guarantee the stability of the closed-loop system with an adaptive nonlinear controller. Numerical simulations are performed to verify the performance of the adaptive nonlinear controller with the proposed slack variable generation method for a quadrotor unmanned aerial vehicle and an unmanned helicopter.

Keywords: Adaptive Control, Underactuated System, Slack Variables

1. INTRODUCTION

There exist various underactuated systems including underwater robots, UAVs(Unmanned Aerial Vehicles), and mobile robots, etc. Underactuated systems have fewer control inputs than the degree of freedom, and therefore, control of the underactuated systems is much more complex than control of fully-actuated systems. For this reason, various nonlinear control methods for the underactuated systems have been developed in the past decades.

Feedback linearization and sliding mode control methods have been widely used to control nonlinear systems. Pathak *et al.* analyzed wheeled inverted pendulum systems to utilize a partial feedback linearization method (Pathak et al. [2005]). Benallegue *et al.* applied a feedback linearization-based controller using high-order sliding mode observer to a quadrotor UAV (Benallegue et al. [2008]). Xu and Ozguner proposed a sliding mode control design approach to stabilize a class of underactuated systems and apply to a translation oscillator with rotational actuator system and a quadrotor UAV (Xu and Ozguner [2008]). A sliding mode control scheme using linear matrix inequality approach has been also used to design controllers for continuous time Markovian jump singular systems with unmeasured states (Wu et al. [2010]) and for differential linear repetitive processes with unmeasurable components of process states and pass profile (Wu et al. [2011]). However, the feedback linearization method only guarantees the local stability of the system which does not consider the disturbance or modeling error. And it is known that the sliding mode control method has some disadvantages such as large control inputs and chattering.

An adaptive nonlinear control scheme has been studied to overcome the demerits of the feedback linearization and the sliding mode control methods. Padhi *et al.* proposed a neural networks-based model-following adaptive control technique for a class of nonaffine nonsquare nonlinear systems (Padhi et al. [2007]). Ahn *et al.* proposed an adaptive sliding mode control method for nonaffine and nonsquare nonlinear systems (Ahn et al. [2007]). An adaptive sliding mode controller was designed for a quadrotor UAV system with the external disturbance (Lee et al. [2009]). And, an adaptive sliding mode controller was integrated to an adaptive image-based visual servoing for a quadrotor UAV system (Lee et al. [2012]). Hong and Kim designed an integrated guidance and controller using adaptive sliding mode control technique for a rotary UAV system (Hong and Kim [2012]). On the other hand, slack variable vectors were used to deal with the properties of nonsquare nonlinear systems. By introducing the slack variables, various adaptive nonlinear control schemes can be used to design a controller for the nonsquare nonlinear systems. However, a systematic slack variable generation method has not been studied much, and only carefully chosen constant slack variable vectors were used to design the adaptive nonlinear controllers (Padhi et al. [2007], Ahn et al. [2007], Lee et al. [2009], Lee et al. [2012], Hong and Kim [2012]).

In this study, a systematic slack variable generation method using QR decomposition is proposed to deal with underactuated nonlinear systems. The proposed slack variable generation method is utilized for an adaptive nonlinear control technique of affine underactuated systems, because the analysis on the affine systems is relatively easier than that of nonaffine systems. QR decomposition is widely used to calculate a solution of a

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least square problem because QR decomposition is known as an efficient and accurate algorithm. QR decomposition can be also utilized to treat the underdetermined systems.

This paper is organized as follows. In Section II, an adaptive sliding mode controller for affine and underactuated nonlinear systems is presented. In Section III, the slack variable generation method using QR decomposition is proposed. In Section IV, an adaptive sliding mode controller and the proposed slack variable generation method is applied to design controllers for a quadrotor UAV system and a unmanned helicopter system. Numerical simulations are performed to verify the performance of the adaptive nonlinear control scheme with the proposed slack variable generation method in Section V. Finally, conclusions are made in Section VI.

2. ADAPTIVE SLIDING MODE CONTROLLER FOR AFFINE UNDERACTUATED SYSTEMS

Consider a control affine underactuated nonlinear system,

$$\dot{\text{x}} = f(\mathbf{x}) + g(\mathbf{x})\mathbf{u} + f_r(\mathbf{x}) \tag{1}$$

where $\mathbf{x} \in \Re^n$ is a state vector, $\mathbf{u} \in \Re^m$ $(m < n)$ is a control vector of the system, and $f_r(\mathbf{x}) \in \Re^n$ is the unknown vector such as the external disturbance or modeling error. Let us assume that the order of system is known, and $f(\mathbf{x})$ and $g(\mathbf{x})$ are continuously differentiable. It is also assumed that $g(\mathbf{x})$ has full rank.

The augmented nonlinear system using slack variables can be written as

$$\dot{\mathbf{x}} = f(\mathbf{x}) + G(\mathbf{x})\mathbf{U} - \nu + f_r(\mathbf{x}) \tag{2}$$

where

$$G(\mathbf{x}) = [g(\mathbf{x}) \;\; \mathbf{g}_s] \tag{3}$$

$$\mathbf{U} = \left[\mathbf{u}^T \;\; \mathbf{u}_s^T\right]^T \tag{4}$$

$$\nu \equiv \mathbf{g}_s \mathbf{u}_s \tag{5}$$

In the above augmented nonlinear system, $\mathbf{g}_s$ is a slack variable matrix to make $G(\mathbf{x})$ invertible, $\mathbf{u}_s \in \Re^{n-m}$ is a slack variable input vector, and ν is the unknown vector which needs to be estimated. Let us consider the assumption that ν and $f_r(\mathbf{x})$ change very slow, respectively.

Consider the following sliding surface,

$$\mathbf{S} \equiv \mathbf{e} = \mathbf{x} - \mathbf{x}_d \tag{6}$$

where $\mathbf{e}$ is the error state vector with respect to the desired state vector, $\mathbf{x}_d$.

Let us define the Lyapunov candidate function as follows

$$L = \frac{1}{2}\mathbf{S}^T\mathbf{S} + \frac{1}{2}\tilde{\nu}^T\Gamma\tilde{\nu} + \frac{1}{2}\tilde{f}_r(\mathbf{x})^T\Omega\tilde{f}_r(\mathbf{x}) \tag{7}$$

where $\tilde{\nu}$ and $\tilde{f}_r(\mathbf{x})$ are the error vectors with respect to the true values of ν and $f_r(\mathbf{x})$, respectively, i.e. $\tilde{\nu} = \nu - \hat{\nu}$ and $\tilde{f}_r(\mathbf{x}) = f_r(\mathbf{x}) - \hat{f}_r(\mathbf{x})$, and Γ and Ω are positive definite diagonal weighting matrices, respectively. Note that $\hat{\nu}$ and $\hat{f}_r(\mathbf{x})$ are the estimated values of ν and $f_r(\mathbf{x})$, respectively.

Let us consider the following control input.

$$\mathbf{U} = G^{-1}(\mathbf{x})\left[-f(\mathbf{x}) + \hat{\nu} - \hat{f}_r(\mathbf{x}) + \dot{\mathbf{x}}_d - C\mathbf{S}\right] \tag{8}$$

with the adaptation rules

$$\dot{\hat{\nu}} = -\Gamma^{-1}\mathbf{S} \tag{9}$$

$$\dot{\hat{f}}_r(\mathbf{x}) = \Omega^{-1}\mathbf{S} \tag{10}$$

where C is a positive definite diagonal gain matrix. Note that the $\dot{\tilde{\nu}}$ and $\dot{\tilde{f}}_r(\mathbf{x})$ have almost same values of $-\dot{\hat{\nu}}$ and $-\dot{\hat{f}}_r(\mathbf{x})$, respectively, since it is assumed that ν and $f_r(\mathbf{x})$ change very slowly. Also, note that the following second order equation can be obtained (Kim et al. [2012]).

$$\ddot{\mathbf{S}} + C\dot{\mathbf{S}} + \left(\Gamma^{-1} + \Omega^{-1}\right)\mathbf{S} = 0 \tag{11}$$

Therefore, it can be stated that the diagonal elements of $\left(\Gamma^{-1} + \Omega^{-1}\right)$ and C have the similar physical meaning on the natural frequency and the damping ratio, respectively.

$$\omega_i^2 = diag\left(\Gamma^{-1} + \Omega^{-1}\right) \tag{12}$$

$$2\zeta_i\omega_i = diag(C) \tag{13}$$

The following augmented control input of (8) with adaptation laws (9) and (10) makes the error states of a control affine underactuated nonlinear system converge to zero, as time increases (Kim et al. [2012]).

Note that the actual control input vector is extracted as

$$\mathbf{u} = \mathbf{U}(\mathbf{1}:\mathbf{m}) \tag{14}$$

The above actual control input is used for the affine underactuated nonlinear system (1).

3. SLACK VARIABLES GENERATION METHOD USING QR DECOMPOSITION

It is very important to choose proper slack variables to make $G(\mathbf{x})$ nonsingular. The conventional approach to select the slack variable is not systematic. RREF(Reduced Row Echelon Form) was utilized to generate slack variables, but the generated slack variables were constant and composed of 0 and 1 due to the properties of RREF (Kim et al. [2012]). In this study, the method generating a suitable slack variable matrix using QR decomposition is proposed. QR decomposition is one of the most widely used methods to calculate a solution of a least square problem. Among various matrix decomposition techniques, QR decomposition is known as a fast and accurate method to compute the least square solutions, and therefore can deal with underdetermined systems, i.e. underactuated systems (Junkins and Kim [1993]).

Theorem 1. Consider a $(n \times m)$ matrix $\mathbf{A}$ that has a full rank m, i.e.,

$$\mathbf{A} \in \Re^{n\times m}, \;\; (n > m), \;\; \rho(\mathbf{A}) = m \tag{15}$$

where $\rho(\cdot)$ denotes the rank of a matrix. Using QR decomposition, $\mathbf{A}$ can be rewritten uniquely as

$$\mathbf{A} = \mathbf{QR} \tag{16}$$

where $\mathbf{Q} \in \Re^{n\times n}$ is an orthonormal matrix, and $\mathbf{R} \in \Re^{n\times m}$ is an upper triangular matrix with positive elements.

The matrix $\mathbf{A}$ can be rewritten due to its nonsquare properties as follows

$$\mathbf{A} = [\mathbf{Q}_1|\mathbf{Q}_2]\begin{bmatrix}\mathbf{R}_1\\ \mathbf{0}\end{bmatrix} \tag{17}$$

where $\mathbf{Q}_1 \in \Re^{n\times m}$, $\mathbf{Q}_2 \in \Re^{n\times(n-m)}$, and $\mathbf{R}_1 \in \Re^{m\times m}$.

Then, $[\mathbf{A}|\mathbf{Q}_2]$ is a nonsingular square matrix, i.e. $\rho([\mathbf{A}|\mathbf{Q}_2]) = n$.

Proof. The proof of Theorem 1 is omitted, since it can be proved easily using the fundamentals of the QR decomposition. (See the proof in Appendix.)