Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 196 (2007) 2413–2430
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

Time dependent subscales in the stabilized finite element approximation of incompressible flow problems

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Received 2 June 2006; received in revised form 26 December 2006; accepted 3 January 2007

Abstract

In this paper we analyze a stabilized finite element approximation for the incompressible Navier–Stokes equations based on the subgrid-scale concept. The essential point is that we explore the properties of the discrete formulation that results allowing the subgrid-scales to depend on time. This apparently "natural" idea avoids several inconsistencies of previous formulations and also opens the door to generalizations.

Keywords: Incompressible flows; Stabilized finite elements; Transient subscales

1. Introduction

Let us start by writing the incompressible Navier–Stokes equations. Consider a domain Ω in $\mathbb{R}^d$, where $d = 2$ or 3 is the number of space dimensions, with boundary $\Gamma = \partial\Omega$, in which we want to solve an incompressible flow problem in the time interval $[0, T]$. If $\boldsymbol{u}$ is the velocity of the fluid and p the pressure, the incompressible Navier–Stokes equations are

$$\partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega,\ t \in]0, T[, \tag{1}$$

$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega,\ t \in]0, T[, \tag{2}$$

where ν is the kinematic viscosity and $\boldsymbol{f}$ is the force vector. These equations must be supplied with an initial condition of the form $\boldsymbol{u} = \boldsymbol{u}^0$ in $\Omega, t = 0$, and a boundary condition which, for simplicity, will be taken as $\boldsymbol{u} = \boldsymbol{0}$ on $\Gamma, t \in]0, T[$.

Let us introduce some standard notation. The space of functions whose p power ($1 \leqslant p < \infty$) is integrable in a domain Ω is denoted by $L^p(\omega)$, $L^\infty(\omega)$ being the space of bounded functions in Ω. The space of functions whose distributional derivatives of order up to $m \geqslant 0$ (integer) belong to $L^2(\omega)$ is denoted by $H^m(\omega)$. The space $H_0^1(\omega)$ consists of functions in $H^1(\omega)$ vanishing on $\partial\omega$. The topological dual of $H_0^1(\omega)$ is denoted by $H^{-1}(\omega)$. A bold character is used to denote the vector counterpart of all these spaces.

If f and g are functions (or distributions) such that fg is integrable in the domain ω under consideration, we denote

$$\langle f, g\rangle_\omega = \int_\omega fg\,\mathrm{d}\omega,$$

so that, in particular, $\langle\cdot,\cdot\rangle_\omega$ is the duality pairing between $H^{-1}(\omega)$ and $H_0^1(\omega)$. When $f, g \in L^2(\omega)$, we write the inner product as $\langle f, g\rangle_\omega \equiv (f, g)_\omega$. The norm in a Banach space X is denoted by $\|\cdot\|_X$, and $L^p(0, T; X)$ is the space of time dependent functions such that their X-norm is $L^p(0, T)$. This notation is simplified in some cases as follows: $(\cdot,\cdot)_\Omega \equiv (\cdot,\cdot)$, $\langle\cdot,\cdot\rangle_\Omega \equiv \langle\cdot,\cdot\rangle$ and $\|\cdot\|_{L^2(\Omega)} \equiv \|\cdot\|$.

Using this notation, the velocity and pressure finite element spaces for the continuous problem are $\boldsymbol{L}^2(0, T; \mathscr{V}_0^d)$ and $L^1(0, T; \mathscr{Q}_0)$ (for example), respectively, where $\mathscr{V}_0^d := \boldsymbol{H}_0^1(\Omega)$, $\mathscr{Q}_0 := L^2(\Omega)/\mathbb{R}$. The weak form of the problem consists in finding $[\boldsymbol{u}, p] \in \boldsymbol{L}^2(0, T; \mathscr{V}_0^d) \times L^1(0, T; \mathscr{Q}_0)$ such that

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doi:10.1016/j.cma.2007.01.002

$$(\partial_t \boldsymbol{u}, \boldsymbol{v}) + \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + \langle \boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v}\rangle - (p, \nabla\cdot\boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle, \quad (3)$$
$$(q, \nabla\cdot\boldsymbol{u}) = 0 \quad (4)$$

for all $[\boldsymbol{v}, q] \in \mathscr{V}_0^d \times \mathscr{Q}_0$, and satisfying the initial condition in a weak sense.

The Galerkin finite element approximation of problem (3), (4) consists in seeking the unknowns in finite dimensional spaces $\mathscr{V}_{0,h}^d \subset \mathscr{V}_0^d$ and $\mathscr{Q}_{0,h} \subset \mathscr{Q}_0$ and taking the test functions also in these spaces. Using the method of lines, the problem discretized in space, but still continuous in time, consists in finding $[\boldsymbol{u}_h(t), p_h(t)] \in \boldsymbol{L}^2(0,T;\mathscr{V}_{0,h}^d)\times L^1(0,T;\mathscr{Q}_{0,h})$ such that

$$(\partial_t \boldsymbol{u}_h, \boldsymbol{v}_h) + \nu(\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) + \langle \boldsymbol{u}_h\cdot\nabla\boldsymbol{u}_h, \boldsymbol{v}_h\rangle - (p_h, \nabla\cdot\boldsymbol{v}_h)$$
$$= \langle \boldsymbol{f}, \boldsymbol{v}_h\rangle, \quad (5)$$
$$(q_h, \nabla\cdot\boldsymbol{u}_h) = 0 \quad (6)$$

for all $[\boldsymbol{v}_h, q_h] \in \mathscr{V}_{0,h}^d \times \mathscr{Q}_{0,h}$.

Once discretized in time (using for example a finite difference scheme), it is well known that problem (5), (6) suffers from different types of numerical instabilities. Two of them are inherited from the stationary problem, namely, the dominance of the (nonlinear) convective term over the viscous one when ν is small and the compatibility required for the velocity and pressure finite element spaces posed by the inf–sup condition. There are also numerical instabilities encountered when the time step size of the time discretization is small, particularly in early stages of the time integration.

A vast literature exists dealing with the instabilities due to the dominance of convection and to the velocity–pressure compatibility condition. In this work we adopt a *stabilized* finite element formulation based on the subgrid-scale concept and, in particular, in the approach introduced by Hughes in [24,26] for the scalar convection–diffusion equation. The basic idea is to approximate the *effect* of the component of the continuous solution which cannot be resolved by the finite element mesh, which we will call *subscale*, on the discrete finite element solution. This approach is a general framework in which it is possible to design different stabilized formulations. We will restrict our attention to two approaches, described in [10,11]. In the first case, the velocity and pressure subscales are taken proportional to the residual of the finite element component in the momentum and in the continuity equations, respectively. The bottom line of the second approach is to take only the component of these residuals L^2 orthogonal to the finite element space. This idea was first introduced in [8] as an extension of a stabilization method originally introduced for the Stokes problem in [12] and fully analyzed for the stationary Navier–Stokes equations in [13].

However, the main interest of this paper is *not* how to stabilize convection-dominated flows or how to be able to use equal velocity–pressure interpolation, thus avoiding the need to satisfy the inf–sup condition that problem (5), (6) demands. Our objective in this paper is *to analyze the formulation that stems from considering time dependent subscales*. In fact, the idea we will follow is not new, and was already introduced in [11]. In this sense, the present work can be considered as a continuation of this reference, with the emphasis placed solely on the consequences of taking the subscales time dependent. We contribute here with the study of several properties of the formulation, including an analysis of its stability and more numerical experiments to check its performance.

The paper is organized as follows. The numerical formulation is described in Section 2, and its main features and its stability analysis are presented in Sections 3 and 4, respectively. In the former, we detail the benefits of considering the subscales time dependent, and how some of the misbehaviors of classical stabilized finite element methods are overcome. We also end Section 3 with a speculative subsection considering the tracking of subscales along the nonlinear process as a way to model turbulence. This idea was also pointed out in [11]. The stability analysis of Section 4 is done for the *linearized* problem, that is, replacing the advection velocity $\boldsymbol{u}$ by a known velocity $\boldsymbol{a}$, which is assumed to be constant. In spite of this simplification, this stability analysis allows us to highlight the improvement in the stability of the original Galerkin formulation (5), (6) introduced by the time dependent subscales. In Section 5 we present the results of three simple numerical examples that show the benefits of our approach. The paper concludes with some final remarks in Section 6.

2. Stabilized finite element problem

Let us consider a finite element partition of the domain Ω with $n_{\rm el}$ elements. A generic element domain will be denoted by K and its diameter by h_K. To simplify the discussion, we will consider quasi-uniform finite element partitions, so that if $h = \max_K h_K$ and $\varrho = \min_K \varrho_K$, with ϱ_K the diameter of the ball inscribed in K, the quotient h/ϱ remains bounded for all partitions. Likewise, we will assume that *all the finite element spaces constructed are continuous* and of the same order for the velocity and the pressure.

The starting idea of the formulation we propose is the variational multiscale formulation proposed in [24,26]. Let $\mathscr{V}_0^d = \mathscr{V}_{0,h}^d \oplus \widetilde{\mathscr{V}}_0^d$, where $\mathscr{V}_{0,h}^d$ is the velocity finite element space and $\widetilde{\mathscr{V}}_0^d$ any space to complete $\mathscr{V}_{0,h}^d$ in $\mathscr{V}_0^d$. Similarly, let $\mathscr{Q}_0 = \mathscr{Q}_{0,h} \oplus \widetilde{\mathscr{Q}}_0$. The original continuous problem (3), (4) is equivalent to find $[\boldsymbol{u}_h(t), p_h(t)] \in \boldsymbol{L}^2(0,T;\mathscr{V}_{0,h}^d)\times L^1(0,T;\mathscr{Q}_{0,h})$, as well as $[\tilde{\boldsymbol{u}}(t), \tilde{p}(t)] \in \boldsymbol{L}^2(0,T;\widetilde{\mathscr{V}}_0^d)\times L^1(0,T;\widetilde{\mathscr{Q}}_0)$, such that

$$(\partial_t(\boldsymbol{u}_h+\tilde{\boldsymbol{u}}), \boldsymbol{v}) + \nu(\nabla(\boldsymbol{u}_h+\tilde{\boldsymbol{u}}), \nabla\boldsymbol{v}) + \langle(\boldsymbol{u}_h+\tilde{\boldsymbol{u}})\cdot\nabla(\boldsymbol{u}_h+\tilde{\boldsymbol{u}}), \boldsymbol{v}\rangle$$
$$- (p_h+\tilde{p}, \nabla\cdot\boldsymbol{v}) = \langle\boldsymbol{f}, \boldsymbol{v}\rangle, \quad (7)$$
$$(q, \nabla\cdot(\boldsymbol{u}_h+\tilde{\boldsymbol{u}})) = 0 \quad (8)$$

for all $[\boldsymbol{v}, q] \in \mathscr{V}_0^d \times \mathscr{Q}_0$. These equations can be split into two systems by taking first $[\boldsymbol{v}, q] = [\boldsymbol{v}_h, q_h] \in \mathscr{V}_{0,h}^d \times \mathscr{Q}_{0,h}$ and then $[\boldsymbol{v}, q] = [\tilde{\boldsymbol{v}}, \tilde{q}] \in \widetilde{\mathscr{V}}_0^d \times \widetilde{\mathscr{Q}}_0$. Denoting by $\boldsymbol{n}$ the