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On the numerical Green's function technique for cracks in Reissner's plates

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Abstract

The numerical Green's function (NGF) technique, previously proposed by the present authors, is here extended to fracture mechanics problems involving Reissner's plate theory. The technique numerically produces a plate bending fundamental Green's function that automatically includes embedded cracks to be used in the classical boundary element method (BEM) to solve this class of problems. The applications discussed include torsion, bending moment and shear force loadings. In addition, also presented is a series of numerical results computed in terms of normalized stress intensity factors to illustrate the good accuracy of the procedure.

Keywords: BEM; LEFM; Reissner theory; Hyper-singular formulation; Cracked plate bending; Green's function

1. Introduction

The Linear Elastic Fracture Mechanics theory employs stresses as well as rotations and dislocations, in the vicinity of the crack tip, to obtain reliable stress intensity factor coefficients, essential in the prediction of tip behaviour and crack stability. There exist many techniques to model cracks using the boundary element method (BEM), these mainly differ in the numerical procedure adopted to include the cracks into the formulation [9]. Sub-regions discretize the crack as the continuation of a fictitious interface using boundary elements for this. The so called dual and displacement discontinuity procedure avoid the interface discretization but keep the use of boundary elements to represent the existing crack. The Green's function approach, also including the adopted numerical Green's function (NGF) treated here, avoids boundary elements over the crack surfaces, since the fundamental solution removes boundary integration there. Concerning the NGF technique, some solutions have been presented in the literature for plates, as in Telles et al. [8] and, recently, in Silveira et al. [6] for crack propagation, but all restricted to in-plane loadings. In the present work, the ability of the hyper-singular formulation to represent a displacement discontinuity is used to develop a NGF solution for Reissner's plate [5,10], i.e., for plates with out of plane loadings also taking into account the transversal shear deformation. The numerical results presented here include bending moment, torsion and shear force loadings to confirm the good accuracy of the procedure.

2. The BEM applied to Reissner's plate theory

The BEM integral equation for generalized displacements (i.e. rotations and linear out of plane displacement) in a Reissner's plate of boundary Γ is [3]

$$C_{ij}(\xi)u_j(\xi) = \int_{\Gamma}\left(u^*_{ij}(\xi,x)p_j(x) - p^*_{ij}(\xi,x)u_j(x)\right)\mathrm{d}\Gamma(x) + \int_{\Omega}\left(u^*_{i3}(\xi,x) - \frac{v}{(1-v)\lambda^2}u^*_{i\alpha,\alpha}(\xi,x)\right)q(x)\,\mathrm{d}\Omega(x), \tag{1}$$

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where ξ and x are the source and field points, respectively; $⨍$ stands for Cauchy's principal value, $C_{ij}(\xi)$ is a geometric coefficient at ξ, equal to δ_{ij} or $\delta_{ij}/2$ for either an internal point ξ or a point on a smooth part of the boundary, $u_{j(x)}$ and $p_{j(x)}$ stand for generalized displacements (two rotations and one deflection), and generalized tractions (moment, torsion and shear tractions), respectively; the fundamental solution, $u^*_{ij}(\xi,x)$ and $p^*_{ij}(\xi,x)$, is the standard [10] infinite domain for Reissner's plates considering a unit point load applied in i direction at ξ; v is Poisson's ratio and λ is the Reissner constant, $\lambda = \sqrt{10}/h$, where h is the thickness of the plate. Throughout the paper, the Greek and the Latin indexes vary from 1–2 and 1–3, respectively.

The domain integral of Eq. (1) represents the transversal uniform distributed loading, $q(x)$, over the domain Ω of the plate. This domain term is not included in what follows (i.e. $q(x) = 0$ is assumed from now on) since it does not interfere with the numerical Green's function evaluation and can be included as a future development in the final formulation.

Suppose a crack as a boundary cavity in Eq. (1), denoted by $\Gamma^F = \Gamma^+ \cup \Gamma^-$ (+ and − standing for "upper" and "lower" surfaces of the crack). Consider a null generalized tractions over the crack boundary or, at the most, $p_j(x^+) = -p_j(x^-)$, what makes null any integrand multiplied by the component $(p_j(x^+) + p_j(x^-))$. Writing the integrals only over boundary Γ^-, the classical and hypersingular formulations, derived from Eq. (1) for $\xi \in \Omega$ are expressed as [2]

$$u_i(\xi) = \int_{\Gamma^-} p^*_{ij}(\xi,x)c_j(x)\,\mathrm{d}\Gamma(x), \tag{2a}$$

$$p_i(\xi) = \int_{\Gamma^-} P^*_{ij}(\xi,x)c_j(x)\,\mathrm{d}\Gamma(x), \tag{2b}$$

where $c_j(x) = u_j(x^+) - u_j(x^-)$ are the generalized crack openings, i.e., the discontinuities in rotations and in deflection at x, and $P^*_{ij}(\xi,x)$ are the properly manipulated derivatives of the standard fundamental solution [3], obtained following the traction boundary integral procedure, expressed as

$$\begin{aligned} P^*_{\alpha\gamma} = \frac{D(1-v)}{4\pi r^2}\Big\{ &(4A(z) + 2zK_1(z) + 1 - v)(\delta_{\alpha\gamma}n_\beta m_\beta + n_\alpha m_\gamma) \\ &+(4A(z) + 1 + 3v)n_\gamma m_\alpha - (16A(z) + 6zK_1(z) + z^2K_0(z) + 2 - 2v) \\ &\times[(-n_\alpha r_{,m} + n_\beta m_\beta r_{,\alpha})r_{,\gamma} + (-\delta_{\alpha\gamma}r_{,m} + m_\gamma r_{,\alpha})r_{,n}] \\ &-2(8A(z) + 2zK_1(z) + 1 + v)(m_\alpha r_{,\gamma}r_{,n} - n_\gamma r_{,\alpha}r_{,m}) \\ &-4(24A(z) + 8zK_1(z) + z^2K_0(z) + 2 - 2v)r_{,\alpha}r_{,\gamma}r_{,n}r_{,m}\Big\}, \end{aligned} \tag{3a}$$

$$\begin{aligned} P^*_{\alpha 3} = \frac{D(1-v)\lambda^2}{4\pi r}\big\{&(2A(z) + zK_1(z))(-n_\alpha r_{,m} + n_\beta m_\beta r_{,\alpha}) \\ &+2(4A(z) + zK_1(z))r_{,\alpha}r_{,m}r_{,n} + 2A(z)m_\alpha r_{,n}\big\}, \end{aligned} \tag{3b}$$

$$\begin{aligned} P^*_{3\gamma} = -\frac{D(1-v)\lambda^2}{4\pi r}\big\{&(2A(z) + zK_1(z))(m_\gamma r_{,n} + n_\beta m_\beta r_{,\gamma}) \\ &+2(4A(z) + zK_1(z))r_{,\gamma}r_{,m}r_{,n} - 2A(z)n_\gamma r_{,m}\big\}, \end{aligned} \tag{3c}$$

$$P^*_{33} = -\frac{D(1-v)\lambda^2}{4\pi r^2}\big\{(z^2B(z) + 1)n_\beta m_\beta + (z^2A(z) + 2)r_{,m}r_{,n}\big\} \tag{3d}$$

in which, $D = \frac{Eh^3}{12(1-v^2)}$ is the flexural rigidity of the plate, E is the Young modulus; only in Eq. (3), n represents the direction of the outward boundary normal vector at x and n_α its components, m is the normal vector direction at the source point ξ and m_α its components; $A(z) = K_0(z) + \frac{2}{z}\left(K_1(z) - \frac{1}{z}\right)$ and $B(z) = K_0(z) + \frac{1}{z}\left(K_1(z) - \frac{1}{z}\right)$ are dependent on the modified Bessel's functions of the second kind $K_0(z)$ and $K_1(z)$, where $z = \lambda r$ and $r = \sqrt{r_\alpha \cdot r_\alpha}$ is the known distance from point x and ξ with components $r_\alpha = x_\alpha(x) - x_\alpha(\xi)$; the derivatives of r are $r_{,\alpha} = \frac{\partial r}{\partial x_\alpha} = \frac{r_\alpha}{r}$, $r_{,n} = \frac{\partial r}{\partial \boldsymbol{n}(x)} = r_{,\alpha}n_\alpha$ and $r_{,m} = \frac{\partial r}{\partial \boldsymbol{m}(\xi)} = -r_{,\alpha}m_\alpha$.

3. Numerical Green's function approach

Let us define ζ as the points over a single straight crack surface, $\Gamma^F(\zeta) = \Gamma^+(\zeta) \cup \Gamma^-(\xi)$, embedded in an infinite medium under the action of a unit point load applied at ξ ($\xi \notin \Gamma^F$). The fundamental solution for this problem at x ($x \notin \Gamma^F$), also called the Green's function, can be written in terms of a superposition of the classical fundamental solution plus a complementary part, so that the sum of both provides satisfaction of the traction free condition on Γ^{F} [8]

$$\begin{aligned} u^{\mathrm{G}}_{ij}(\xi,x) &= u^*_{ij}(\xi,x) + u^{\mathrm{c}}_{ij}(\xi,x), \\ p^{\mathrm{G}}_{ij}(\xi,x) &= p^*_{ij}(\xi,x) + p^{\mathrm{c}}_{ij}(\xi,x). \end{aligned} \tag{4}$$

Superscripts G, ∗ and c, stand for Green's fundamental solution, classical Reissner's plate solution and complementary components of the respective fundamental displacements and tractions. Subscript i stands for the direction of the applied load at ξ and j for the respective component at x ($x \notin \Gamma^F$). Using Eq. (2), the complementary solutions are expressed by the following integrals:

$$u^{\mathrm{c}}_{ij}(\xi,x) = \int_{\Gamma^-} p^*_{jk}(x,\zeta)c_{ik}(\xi,\zeta)\,\mathrm{d}\Gamma(\zeta) \tag{5}$$

$$p^{\mathrm{c}}_{ij}(\xi,x) = \int_{\Gamma^-} P^*_{jk}(x,\zeta)c_{ik}(\xi,\zeta)\,\mathrm{d}\Gamma(\zeta) \tag{6}$$

defining the complementary generalized displacements and tractions at an internal point x ($x \notin \Gamma^F$), as a function of the generalized fundamental displacement discontinuities $c_{ik}(\xi,\zeta) = u_{ik}(\xi,\zeta^+) - u_{ik}(\xi,\zeta^-)$ associated to a unit point load at ξ. Note that the fundamental $c_{ik}(\xi,\zeta)$ is the solution of the complementary problem in which the load at the crack line is minus the standard full space tractions for a unit point load at ξ. Eqs. (5) and (6) can be numerically evaluated using Gaussian quadrature to obtain the fundamental Green's functions as

$$\begin{aligned} u^{\mathrm{G}}_{ij}(\xi,x) &= u^*_{ij}(\xi,x) + \sum_{n=1}^{N} p^*_{jk}(x,\zeta_n)c_{ik}(\xi,\zeta_n)J_nW_n, \\ p^{\mathrm{G}}_{ij}(\xi,x) &= p^*_{ij}(\xi,x) + \sum_{n=1}^{N} P^*_{jk}(x,\zeta_n)c_{ik}(\xi,\zeta_n)J_nW_n, \end{aligned} \tag{7}$$

where J_n is the Jacobian of the transformation to the standard quadrature interval, W_n is the corresponding weight factor at the Gauss station n, N is the total number