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Hurwitz stability analysis of fractional order LTI systems according to principal characteristic equations

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ABSTRACT

With power mapping (conformal mapping), stability analyses of fractional order linear time invariant (LTI) systems are carried out by consideration of the root locus of expanded degree integer order polynomials in the principal Riemann sheet. However, it is essential to show the left half plane (LHP) stability analysis of fractional order characteristic polynomials in the s plane in order to close the gap emerging in stability analyses of fractional order and integer order systems. In this study, after briefly discussing the relation between the characteristic root orientations and the system stability, the author presents a methodology to establish principal characteristic polynomials to perform the LHP stability analysis of fractional order systems. The principal characteristic polynomials are formed by factorizing principal characteristic roots. Then, the LHP stability analysis of fractional order systems can be carried out by using the root equivalency of fractional order principal characteristic polynomials. Illustrative examples are presented to explain how to find equivalent roots of fractional order principal characteristic polynomials in order to carry out the LHP stability analyses of fractional order nominal and interval systems.

1. Introduction

Asymptotical stability of LTI systems is commonly decided by analyzing the root locus of the system characteristic polynomials. Specifically, the case of placement of all roots of characteristic polynomials on the left hand side of the complex s plane has been recognized as a clear indication of asymptotically settling of system outputs because it infers the exponential decay of the amplitude of transient components in the analytical time domain solutions. Such asymptotically stable LTI systems are also known as Hurwitz stable or left half plane (LHP) stable systems. For control theory, it is essential to demonstrate the LHP stability of fractional order systems in s plane so that it can reduce methodological discrepancies in analyses of integer and fractional order systems. These analyses may offer deeper insight on the behavior of fractional order systems.

For stability analyses of fractional order systems, fractional order characteristic polynomials are transformed into the expanded degree integer order characteristic polynomials by applying $v = s^{1/m}$ mapping. Afterward, roots of these expanded degree integer order characteristic polynomials are calculated, and stability analyses of fractional order systems are carried out in the first (principal) Riemann sheet according to root placements [1–8]. Power mapping of polynomials, also known as conformal mapping, was discussed in detail [2,6,9]. The use of $v = s^{1/m}$ mapping for stability analysis was shown in [1,2,6], and examples were given in [1,2]. Throughout the current paper, the author prefers to use the phrase "power mapping" for $v = s^{1/m}$ transformation so that the mapping is based on the power arithmetic of complex variables of polynomials. The main advantage of power mapping comes from ease of factorization of polynomials when the fractional order polynomials are transformed to integer order polynomials. On the other hand, $v = s^{1/m}$ power mapping also maps the Hurwitz stability region (HSR), namely the LHP of the complex s plane, into a fragment of the first Riemann sheet of the complex v plane. It was revealed in previous works that only characteristic roots in the first Riemann sheet were meaningful for system stability [1,2]. In the current study, in order to perform Hurwitz stability analysis at the LHP of the s plane, author suggests the need to establish principal characteristic polynomials formed by factorization of characteristic roots in the first Riemann sheet of the v plane. Then, this polynomial is used to convey all information, meaningful for system stability, back to the s plane by applying $s = v^m$ inverse power mapping. Finally, the root equivalency class of polynomials is used to obtain roots of fractional order principal characteristic polynomials in the s plane, and thus LHP stability analyses are achieved.

Over the last few decades, robust stability analysis of fractional order systems has been addressed in many aspects, using methods based on Linear Matrix Inequality (LMI) analysis [5,10–12], methods based on value set analysis and zero exclusion principles

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[12,13,14], methods based on root placement [1–8], and so on. However, LHP stability of fractional order systems has not yet been adequately discussed. Primarily, this study aims to show how to convey root information back to complex s plane for the purpose of LHP stability analysis. Thus, it can be possible to carry out Hurwitz stability analysis in the LHP of s plane, and this may close the methodological gaps appearing in stability analyses of fractional order LTI systems and integer order LTI systems. Illustrative examples are presented to show the LHP stability analyses for fractional order nominal and interval uncertain LTI systems, and the results are discussed.

System analysis for the fractional order polynomials is more complicated than the analysis for integer order polynomials. The renewed efforts to deal with these complications may broaden and refresh our views on systems. The current study demonstrates utilization of two innovative properties in the stability analysis of fractional order systems: (i) root equivalency class and (ii) principal characteristic polynomial. Root equivalency arithmetic defines a class of equivalent characteristic roots of fractional order systems, and stability states are conformal in the root equivalent class of fractional order characteristic polynomials. The other noteworthy point of this work is that one can establish fractional order principal characteristic polynomials that contain information related to system stability. As a consequence, with these properties, it is possible to demonstrate the LHP stability analysis of fractional order systems in a similar manner to the LHP stability analysis of integer order systems in the s plane.

2. Definitions and preliminaries

2.1. Root equivalency class of characteristic polynomials

Let us factorize the integer order characteristic polynomial in the form of $\Delta(s) = \prod_{i=1}^{n}(s - z_i)$. Here, $z_1, z_2, z_3 \ldots, z_n$ are complex roots, where $n \in Z^+$. To have the same characteristic roots, the α order root equivalency class of characteristic polynomials is defined as,

$$\Delta_E = \left\{ \prod_{i=1}^{n} \left(s^{\alpha_i} - z_i^{\alpha_i} \right) \; : \; \alpha_i > 0 \;\; \wedge \;\; n \in Z^+ \right\} \tag{1}$$

Proof of the root equivalency class Δ_E can be given as: The roots of the integer order characteristic equation, $\Delta(s) = \prod_{i=1}^{n}(s - z_i) = 0$, are obtained by solving each factor $s - z_i = 0$ as $s = z_i \in C$. By applying α_i power to the both sides of the solutions $s = z_i$, one obtains $s^{\alpha_i} = z_i^{\alpha_i}$ and writes factors of fractional order polynomials as $s^{\alpha_i} - z_i^{\alpha_i} = 0$. By side to side multiplication of all factors, α_i power root equivalency can be written as $\prod_{i=1}^{n}(s^{\alpha_i} - z_i^{\alpha_i})$. In this case, the set of α order root equivalency class can be written in the form of Eq. (1).

2.2. Riemann sheets in complex domains

Let us consider a fractional order polynomial with real coefficients expressed in the form of $p_f(s) = \sum_{i=0}^{n} c_i s^{\alpha_i}$, where $\alpha_i \in R^+$ for $i = 1, 2, 3, \ldots, n$. In the case of $\alpha_0 = 0$, the coefficient c_0 yields the constant term of polynomials. The $p_f(s)$ is a multi-valued function of the complex variable s, whose domain are Riemann surface of a finite number of sheets for $\forall\, i,\ \alpha_i = Q^+$ [1,2]. In this case, a fractional order α is written $\alpha = 1/m$, where m is a positive integer number and the m sheets of Riemann surface are defined in [1] as, $s = |s|e^{j\theta}$, $(2k+1)\pi < \theta < (2k+3)\pi$, $k = -1, 0, 1, \ldots, m-2$. For $k = -1$, the principal Riemann sheet is expressed as $-\pi < \theta < \pi$ [1,2]. For $v = s^{1/m}$ mapping, Riemann sheets become the regions of the v plane, which are defined in [1] by $v = |v|e^{j\phi}$, $(2k+1)\frac{\pi}{m} < \phi < (2k+3)\frac{\pi}{m}$, $k = -1, 0, 1, \ldots, m-2$. Here, fractional orders α_i can be written in the form of division of two integers as $\alpha_i = \frac{k_i}{h_i}$ (for $k_i = 0$ and $h_i = 1$). A lower bound of m was defined by least common multiple (LCD) of $h_1, h_2, h_3, \ldots, h_n$, which was denote as $m = LCD(h_1, h_2, h_3, \ldots, h_n)$ in [1,2,15,16].

2.3. Hurwitz stability (LHP Stability) of integer order characteristic polynomials

Let us consider an integer order characteristic polynomial in the form of $p(s) = \sum_{i=0}^{n} c_i s^i$. Parameters $c_i \in R$ are coefficients of the real polynomial, and the parameter $n \in Z^+$ represents the degree of the polynomial. A characteristic polynomial with real coefficients is said to be Hurwitz stable if and only if all of its roots lie in the left hand side of the complex s plane [17,18]. The Hurwitz stable polynomial family is described as $\{ p(s) | p(s) = 0 : \forall\, s \in C \wedge \mathrm{Re}\{s\} < 0 \}$, which is also called LHP stability. The condition $\mathrm{Re}\{s\} < 0$ refers to open left half plane of the complex s plane. If the characteristic polynomial of a LTI system model is a Hurwitz stable polynomial, the LTI system model behaves asymptotically stable because time domain solutions consist of exponentially decaying transient terms. Therefore, the root locus of characteristic polynomials has been widely used for asymptotic stability analyses of LTI systems.

In general, characteristic polynomial $p(s)$ is a multi-valued function of the complex variable of s, whose domain was described by the first sheet (principal sheet) of the Riemann surface, defined in an argument range $-\pi < \arg(s) < \pi$ [1]. As is already known, the Hurwitz stability region (HSR) is the left half plane of the first sheet, which can be defined according to root arguments as $\pi/2 < |\arg(s)| \le \pi$. It is also convenient to refer to the argument bounds, $\pi/2$ and $3\pi/2$, as to Hurwitz stability boundary (HSB). The arguments bounds $\pi/2$ and $3\pi/2$ are the angles of the imaginary axis of the complex s plane.

2.4. Hurwitz stable fractional order polynomials

Let us consider a fractional order polynomial $p_f(s)$. The factorization of fractional order polynomials to find out polynomial roots is more complicated than the factorization of integer order polynomials. In order to facilitate root locus analysis of fractional order LTI systems, $v = s^{1/m}$ mapping has been purposed to transform a fractional order characteristic polynomial to the expanded degree integer order characteristic polynomials [1–8], which were defined as,

$$p_f(s)\big|_{s=v^m} = p_m(v) = \sum_{i=0}^{n} c_i v^{m\alpha_i} \tag{2}$$

Here, the term of $m\alpha_i$ for $i = 0, 1, 2.., n$ is an integer number. Following $v = s^{1/m}$ mapping, the first Riemann sheet is confined to a plane slice with the argument range $-\pi/m < \arg(v) < \pi/m$ [1,2], and the stability analyses were carried out in this argument range, which is also known as the first Riemann sheet [1,2,6–8,19,20]. In practice, the roots in the first Riemann sheet are meaningful for stability analysis [1,2]. In related works, after applying $v = s^{1/m}$ mapping, characteristic polynomials were recognized to be stable, in the case that all roots in the first Riemann sheet lie in the complex plane slice with the argument range of $(\pi/2m, \pi/m]$ and $[-\pi/m, -\pi/2m)$ as depicted Fig. 1. Some remarks of power mapping can be given as follows [2,9]:

Remark 1. *(Power mapping of real polynomials)*: Let us consider the fraction order real coefficient polynomial $p_f(s)$, where the input