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Impulsive stabilization of fractional differential systems ☆

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ABSTRACT

This paper investigates the impulsive stabilization problem of fractional differential systems (FDSs in short). Both the global exponential stability and ultimate boundedness criteria are established using Lyapunov functions, algebraic inequality techniques and boundedness of Mittag-Leffler functions. It is shown that unstable and unbounded FDSs can be stable and bounded respectively under impulsive control. Examples and simulations are also provided to demonstrate the effectiveness of the derived theoretical results.

1. Introduction

Fractional differential systems (FDSs in short) are the generalizations of the classical integer order differential systems. In recent years, FDSs have gained an increasing attention due to their potential applications in various areas such as biology, rheology, viscoelasticity, electrochemistry, capacitor theory, control theory, electrical networks, fluid dynamics, porous media, blood flow phenomena, nonlinear oscillation of earthquake, etc. Many significant contributions have been made in the theory of FDSs [1–14].

On the other hand, impulsive effect is one of the most familiar phenomena in many evolution processes in which states exhibit abrupt changes at certain moments, involving many fields such as economics, mechanics, epidemic models, neural networks and satellite communications, etc. In the past few decades, the qualitative and stability theory for different kinds of dynamical systems with impulsive effects have been investigated deeply and widely, and many important results have been achieved [15–20]. Recently, great efforts have been devoted to extend the stability from impulse-free FDSs to impulsive FDSs (IFDSs in short). Up to now, many interesting results on IFDSs have been obtained [21–32]. Most of these results focus only on the stability and existence-uniqueness but the study of boundedness problem of IFDSs is of paramount importance since boundedness plays a key role in investigating the basic properties of solutions. Therefore, it is natural that studies on IFDSs involve not only a discussion of the stability and existence-uniqueness but also boundedness. Unfortunately, little attention has been paid to the boundedness problem of IFDSs. In [33], Xu et al. studied the global ultimate boundedness problem of impulsive Caputo FDSs (ICFDSs in short) under the condtion ${}_{t_{k-1}}^{\;C}D_t^qV(t,y(t)) \le \eta V(t,y(t)) + \Upsilon$ $(t \in (t_{k-1}, t_k])$ with $\eta < 0$ and $\Upsilon \ge 0$. There is no doubt that the results would collapse if $\eta \ge 0$. Then a nature question aries: what are the boundedness criteria for the case of $\eta \ge 0$? In other words, what are the boundedness criteria for IFDSs if the corresponding impulse-free FDSs are unboundedness themselves? To the best of our knowledge, there are no publications dealing with this problem, which remains an interesting and challenging research topic. Therefore, the issue constitutes the main motivation of this paper.

Summarizing the above discussions, our objective in this paper is to discussed the boundednees problem of ICFDSs for the case $\eta \ge 0$, that is, to discuss the impulsive stabilization problem of FDSs. Using Lyapunov functions, algebraic inequality techniques and boundedness of Mittag-Leffler functions, both the global exponential stability and ultimate boundedness criteria are established. It is shown that unstable and unbounded impulse-free FDSs can be stable and bounded respectively under impulsive control. Examples are also given to explain our results.

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2. Preliminaries

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space with norm $\|\cdot\|$, $\mathbb{R}^{n\times n}$ denote the set of $n\times n$-dimensional real matrices, $\mathbb{R}_+$ denote the set of non-negative real numbers, $\mathbb{Z}_+$ denote the set of positive integers, and $\mathbb{R}_{t_0}=[t_0,\infty)$. Let $\mathfrak{C}(\mathbb{R}_{t_0}\times\mathbb{R}^n,\mathbb{R}_+)$ be the collection of all piecewise continuous functions from $\mathbb{R}_{t_0}\times\mathbb{R}^n$ to $\mathbb{R}_+$. For a real symmetric matrix A, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denote the maximum and the minimum eigenvalue of A, respectively. In the following, we recall some definitions of fractional calculus in [6].

Gamma function $\Gamma(z)$:

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt,\ \mathrm{Re}(z)>0$$

where $\mathrm{Re}(z)$ denotes the real part of the complex number z.

The Caputo fractional derivative is defined as

$$ {}_{t_0}^{C}D_t^q y(t)=\frac{1}{\Gamma(n-q)}\int_{t_0}^{t}\frac{y^{(n)}(s)}{(t-s)^{q+1-n}}ds,\ n-1<q<n. \tag{1}$$

The one-parameter and two-parameter Mittag-Leffler functions are defined as, respectively

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)},\ (\alpha>0), \tag{2}$$

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)},\ (\alpha>0,\ \beta>0). \tag{3}$$

Obviously, $E_\alpha(z)=E_{\alpha,1}(z)$.

Consider the following IFDSs:

$$\begin{cases} {}_{t_0}^{C}D_t^q y(t)=Ay(t)+f(t,y(t)), & t\neq t_k,\ t\geq t_0,\\ \Delta y(t_k)=y(t_k^+)-y(t_k^-)=I_k(y(t_k^-)), \\ y(t_0^+)=y_0 \end{cases} \tag{4}$$

where $q\in(0,1)$, $A\in\mathbb{R}^{n\times n}$, $f:[t_0,\infty)\times\mathbb{R}^n\to\mathbb{R}^n$, $y(t_k^+)=\lim_{\epsilon\to0^+}y(t_k+\epsilon)$, $y(t_k^-)=\lim_{\epsilon\to0^-}y(t_k+\epsilon)$, $I_k:\mathbb{R}^n\to\mathbb{R}^n$, and the impulsive moments $t_k(k\in\mathbb{Z}_+)$ are a strictly increasing sequence which satisfies $\lim_{k\to\infty}t_k=\infty$.

Throughout this paper, we assume that for any given $y_0\in\mathbb{R}^n$, there exits at least one solution $y(t)$ of (4), which is left continuous at each t_k, i.e. $y(t_k^-)=y(t_k)$. One may refer to [21,26,31], for the results on the existence and uniqueness of the solution of IFDSs.

Lemma 2.1. *[6] If $0<\alpha<2$, β is an arbitrary real number, ζ is such that $\frac{\pi\alpha}{2}<\zeta<\min\{\pi,\pi\alpha\}$ and C_1, C_2 are real constants, then*

$$|E_{\alpha,\beta}(z)|\leq C_1(1+|z|)^{\frac{1-\beta}{\alpha}}\exp\left(\mathrm{Re}\left(z^{\frac{1}{\alpha}}\right)\right)+\frac{C_2}{1+|z|},(|\arg(z)|\leq\zeta),\ |z|\geq0.$$

Remark 2.1. Using Lemma 2.1 and the nonnegativity of $E_q(\eta(t_i-t_{i-1})^q)$ and $E_{q,q+1}(\eta(t_i-t_{i-1})^q)$, we have there exist positive constants ϑ_1,ϑ_2,ϑ_3 and ϑ_4 such that

$$\begin{aligned} E_q(\eta(t_i-t_{i-1})^q) &\leq \vartheta_1\exp\left(\mathrm{Re}(\eta(t_i-t_{i-1})^q)^{\frac{1}{q}}\right)+\frac{\vartheta_2}{1+|\eta(t_i-t_{i-1})^q|}\\ &\leq \vartheta_1 e^{\eta^{\frac{1}{q}}(t_i-t_{i-1})}+\frac{\vartheta_2}{1+\eta(t_i-t_{i-1})^q}, \end{aligned}\tag{5}$$

$$\begin{aligned} E_{q,q+1}(\eta(t_i-t_{i-1})^q) &\leq \vartheta_3(1+|\eta(t_i-t_{i-1})^q|)^{-1}\exp\left(\mathrm{Re}(\eta(t_i-t_{i-1})^q)^{\frac{1}{q}}\right)\\ &\quad+\frac{\vartheta_4}{1+|\eta(t_i-t_{i-1})^q|}\\ &\leq \vartheta_3 e^{\eta^{\frac{1}{q}}(t_i-t_{i-1})}+\frac{\vartheta_4}{1+\eta(t_i-t_{i-1})^q},\ \forall\, i\in\mathbb{Z}_+,\ \eta\geq0. \end{aligned}\tag{6}$$

Lemma 2.2. *Suppose that there exists a function $V(t,y)\in\mathfrak{C}^{1,2}(\mathbb{R}_{t_0}\times\mathbb{R}^n,\mathbb{R}_+)$ and several constants $\Upsilon\geq0$, $\mu_k>0$, and $\eta\geq0$ such that.*

(1) for all $k\in\mathbb{Z}_+$ and $y\in\mathbb{R}^n$,

$$V(t_k^+,y(t_k^+))\leq\mu_k V(t_k,y(t_k)); \tag{7}$$

(2) for all $t_{k-1}<t\leq t_k$, $k\in\mathbb{Z}_+$, and $y\in\mathbb{R}^n$,

$$ {}_{t_{k-1}}^{C}D_t^q V(t,y(t))\leq\eta V(t,y(t))+\Upsilon. \tag{8}$$

Then all solutions of (4) satisfy the following estimate

$$\begin{aligned} V(t,y(t))\leq&\ \Pi_{i=1}^{k}\mu_i E_q\left(\eta\theta_i^q\right)E_q\left(\eta(t-t_k)^q\right)V(t_0,y(t_0))\\ &+\sum_{j=1}^{k-1}\left\{\Pi_{i=1}^{j}\mu_{i+k-j}E_q\left(\eta\theta_{i+k-j}^q\right)\mu_{k-j}E_q\left(\eta(t-t_k)^q\right)E_{q,q+1}\left(\eta\theta_{k-j}^q\right)\Upsilon\theta_{k-j}^q\right\}\\ &+\mu_k E_q\left(\eta(t-t_k)^q\right)E_{q,q+1}\left(\eta\theta_k^q\right)\Upsilon\theta_k^q+\Upsilon(t-t_k)^q E_{q,q+1}\left(\eta(t-t_k)^q\right),\\ &t\in(t_k,t_{k+1}],\ k\geq0, \end{aligned}\tag{9}$$

where $0<\theta_i=t_i-t_{i-1}<\infty$ and $\Pi_{i=1}^{0}(\cdot)_i=1$, $\sum_{j=1}^{-1}(\cdot)_i=\sum_{j=1}^{0}(\cdot)_i=0$.

Proof. The proof of Lemma 2.2 is similar to that of Inequality (22) in [33].

Lemma 2.3. *([11])Let $y(t)\in\mathbb{R}^n$ be a vector of differentiable function. Then for any time constant $t\geq t_0$, the following relationship holds*

$$ {}_{t_0}^{C}D_t^q(y^T(t)\mathcal{P}y(t))\leq 2y^T(t)\mathcal{P}\,{}_{t_0}^{C}D_t^q y(t), \tag{10}$$

where $q\in(0,1)$, $\mathcal{P}\in\mathbb{R}^{n\times n}$ is a constant, symmetric and positive definite matrix.

Lemma 2.4. *([34]) Let $X\in\mathbb{R}^{n\times n}$ be a positive definite matrix and $Q\in\mathbb{R}^{n\times n}$ a symmetric matrix. Then for any $y\in\mathbb{R}^n$, the following inequality holds*

$$\lambda_{\min}(X^{-1}Q)\cdot y^TXy\leq y^TQy\leq\lambda_{\max}(X^{-1}Q)y^TXy.$$

3. Main results

Theorem 3.1. *Assume that there exists a function $V(t,y)\in\mathfrak{C}(\mathbb{R}_{t_0}\times\mathbb{R}^n,\mathbb{R}_+)$ and several constants $\Upsilon\geq0$, $\mu_k>0$, $c_1>0$, $c_2>0$ and $\eta\geq0$ such that.*

(i) for all $(t,y)\in\mathbb{R}_{t_0}\times\mathbb{R}^n$,

$$c_1\|y\|^2\leq V(t,y)\leq c_2\|y\|^2; \tag{11}$$

(ii) for all $k\in\mathbb{Z}_+$ and $y\in\mathbb{R}^n$,

$$V(t_k^+,y(t_k^+))\leq\mu_k V(t_k,y(t_k)); \tag{12}$$

(iii) for all $t_{k-1}<t\leq t_k$, $k\in\mathbb{Z}_+$, and $y\in\mathbb{R}^n$,

$$ {}_{t_{k-1}}^{C}D_t^q V(t,y(t))\leq\eta V(t,y(t))+\Upsilon; \tag{13}$$

(iv) $0<\theta_k=t_k-t_{k-1}<\infty$, $k\in\mathbb{Z}_+$,

$$\mu\left(\vartheta_1 e^{h\eta^{\frac{1}{q}}}+\frac{\vartheta_2}{1+\eta\theta^q}\right)<1, \tag{14}$$

where $h=\sup_{k\in\mathbb{Z}_+}\{\theta_k\}$, $\mu=\sup_{k\in\mathbb{Z}_+}\{\mu_k\}$ and $\theta=\inf_{k\in\mathbb{Z}_+}\{\theta_k\}$. Then

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