Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 196 (2007) 853–866
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

Stabilized finite element methods for the generalized Oseen problem

M. Braack [a], E. Burman [b], V. John [c], G. Lube [d,*]

[a] Institute of Applied Mathematics, University of Heidelberg, Germany
[b] Institut d'Analyse et de Calcul Scientifique (CMCS/IACS), Ecole Polytechnique Federale de Lausanne, CH-1005 Lausanne, Switzerland
[c] FR 6.1 – Mathematics, University of the Saarland, Saarbrücken, Germany
[d] Institute of Numerical and Applied Mathematics, University of Göttingen, Germany

Received 12 October 2005; received in revised form 15 March 2006; accepted 6 July 2006

Abstract

The numerical solution of the non-stationary, incompressible Navier–Stokes model can be split into linearized auxiliary problems of Oseen type. We present in a unique way different stabilization techniques of finite element schemes on isotropic meshes. First we describe the state-of-the-art for the classical residual-based SUPG/PSPG method. Then we discuss recent symmetric stabilization techniques which avoid some drawbacks of the classical method. These methods are closely related to the concept of variational multiscale methods which seems to provide a new approach to large eddy simulation. Finally, we give a critical comparison of these methods.

Keywords: Incompressible flow; Navier–Stokes equations; Variational multiscale methods; Stabilized finite elements

1. Introduction

The motivation of the present paper stems from the finite element simulation of the incompressible Navier–Stokes problem

$$\partial_t \mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \tilde{\mathbf{f}}, \tag{1}$$
$$\nabla\cdot\mathbf{u} = 0 \tag{2}$$

for the velocity $\mathbf{u}$ and the pressure p in a polyhedral domain $\Omega \subset \mathbb{R}^d$, $d \leqslant 3$, with a given source term $\tilde{\mathbf{f}}$. A standard algorithmic treatment of (1) and (2) is to semidiscretize in time (with possible step length control) using an A-stable method and to apply a fixed point or Newton-type iteration per time step. This leads to the following auxiliary problem of Oseen type in each step of this iteration:

$$L_{\mathrm{Os}}(\mathbf{b};\mathbf{u},p) := -\nu\Delta\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + c\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{3}$$
$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega. \tag{4}$$

Also the iterative solution of the steady state Navier–Stokes equations using a fixed point iteration leads to problems of type (3) and (4) with $c = 0$.

The standard Galerkin finite element method (FEM) for (3) and (4) may suffer from two problems:

- dominating advection (and reaction) in the case of $0 < \nu \ll \|\mathbf{b}\|_{L^\infty(\Omega)}$,
- violation of the discrete inf–sup (or Babuška–Brezzi) stability condition for the velocity and pressure approximations.

The well-known *streamline upwind/Petrov–Galerkin (SUPG) method*, introduced in [5], and the *pressure-stabilization/Petrov–Galerkin* (PSPG) method, introduced in [31,26], opened the possibility to treat both problems in a unique framework using rather arbitrary FE approximations of velocity and pressure, including equal-order pairs. Additionally to the Galerkin part, the elementwise residual $L_{\mathrm{Os}}(\mathbf{b};\mathbf{u},p) - \mathbf{f}$ is tested against the (weighted) non-symmetric part $(\mathbf{b}\cdot\nabla)\mathbf{v} + \nabla q$ of $L_{\mathrm{Os}}(\mathbf{b};\mathbf{v},q)$. Moreover, it was shown in [18,23,40] that an additional element-wise stabilization of the divergence constraint (4), henceforth denoted

* Corresponding author.
E-mail addresses: braack@iwr.uni-heidelberg.de (M. Braack), erik.burman@epfl.ch (E. Burman), john@math.uni-sb.de (V. John), lube@math.uni-goettingen.de (G. Lube).

0045-7825/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.cma.2006.07.011

as *grad–div stabilization*, is important for the robustness if $0 < \nu \ll 1$. Due to its construction, we will classify the SUPG/PSPG/grad–div approach as an (*element-wise*) *residual-based stabilization* technique.

Despite the success of this classical stabilization approach to incompressible flows over the last 20 years, one can find in recent papers a critical evaluation of this approach, see e.g. [20,12]. Drawbacks are basically due to the strong coupling between velocity and pressure in the stabilizing terms. (For a more detailed discussion, cf. Section 7.) Several attempts have been made to relax the strong coupling of velocity and pressure and to introduce symmetric versions of the stabilization terms:

- Recently, the interior penalty technique of the discontinuous Galerkin (DG) method was applied in the framework of continuous approximation spaces as proposed in [17] leading to the *edge/face oriented stabilization* introduced in [12]. It can be classified as well as a residual-based stabilization technique since it controls the inter-element jumps of the non-symmetric terms in (3) and (4).
- Another approach consists in *projection-based stabilization* techniques. The first step was done in [16] where weighted *global* orthogonal projections of the non-symmetric terms in (3) and (4) are added to the Galerkin scheme. A related *local* projection technique has been applied to the Oseen problem in [3] with low-order equal-order interpolation. Another projection-based stabilization was introduced in [32,29].

The projection-based methods are closely related to the framework of *variational multiscale methods* introduced in [25]. The latter method provides a new approach to large eddy simulation (LES) of incompressible flows which does not possess important drawbacks of the classical LES like commutation errors.

The goal of the present paper is a unique presentation of residual-based and projection-based stabilization techniques to the numerical solution of the Oseen problem (3) and (4), together with a critical comparison.

For brevity, we consider only *conforming* FEM. An extension to a non-conforming approach like DG-methods in an element- or patch-wise version can be found, e.g., in [14,21]. The latter methods are not robust with respect to the viscosity ν. An overview of appropriate stabilization mechanisms in the DG framework was given in [4].

The paper is organized as follows: In Section 2, we describe the basic Galerkin discretization of the Oseen problem. Then, we consider residual-based stabilization methods including the classical SUPG/PSPG/grad–div stabilization following [36], see Section 3, and the edge/face-stabilization method following [12,13], see Section 4. Next, we present projection-based stabilization techniques. Here, we review the local projection approach proposed in [3], see Section 5, and another projection-based stabilized scheme due to [32,29], see Section 6. A critical comparison of the schemes can be found in Section 7.

2. The standard Galerkin FEM for the Oseen problem

Throughout this paper, we will use standard notations for Lebesgue and Sobolev spaces. The L^2-inner product in a domain ω is denoted by $(\cdot,\cdot)_\omega$. Without index, the L^2-inner product in Ω is meant.

This section describes the standard Galerkin FEM for the Oseen-type problem (3) and (4), for simplicity of presentation with homogeneous Dirichlet data:

$$L_{\mathrm{Os}}(\mathbf{b};\mathbf{u},p) := -\nu\Delta\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + c\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{5}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \tag{6}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \tag{7}$$

with $\mathbf{b} \in [H^1(\Omega)\cap L^\infty(\Omega)]^d$, ν, $c \in L^\infty(\Omega)$, $\mathbf{f} \in [L^2(\Omega)]^d$ and

$$\nu > 0, \quad (\nabla\cdot\mathbf{b})(x) = 0, \quad c(x) \geqslant c_{\min} \geqslant 0, \quad \text{a.e. in } \Omega. \tag{8}$$

Let $H^1_0(\Omega) := \{v \in H^1(\Omega) \mid v|_{\partial\Omega} = 0\}$ and $L^2_0(\Omega) := \{q \in L^2(\Omega) \mid \int_\Omega q\,\mathrm{d}x = 0\}$. The variational formulation reads: find $U = \{\mathbf{u},p\} \in \mathbf{V}\times\mathbf{Q} := [H^1_0(\Omega)]^d \times L^2_0(\Omega)$ s.t.

$$\mathscr{A}(\mathbf{b};U,V) = \mathscr{L}(V) \quad \forall V = \{\mathbf{v},q\} \in \mathbf{V}\times\mathbf{Q} \tag{9}$$

with

$$\mathscr{A}(\mathbf{b};U,V) = (\nu\nabla\mathbf{u},\nabla\mathbf{v}) + ((\mathbf{b}\cdot\nabla)\mathbf{u} + c\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) - b(\mathbf{u},q), \tag{10}$$

$$\mathscr{L}(V) = (\mathbf{f},\mathbf{v}), \tag{11}$$

$$b(\mathbf{v},p) = -(p,\nabla\cdot\mathbf{v}). \tag{12}$$

Suppose an admissible triangulation $\mathscr{T}_h$ of the polyhedral domain Ω. We assume that $\mathscr{T}_h$ is shape-regular, i.e., there exists a constant C_{sh}, independent of the meshsize h with $h_T = h|_T$, such that $C_{sh}h_T^d \leqslant \mathrm{meas}(T)$ for all $T \in \bigcup_h \mathscr{T}_h$. In particular, we exclude anisotropic elements throughout the paper.

Moreover, we assume that each element $T \in \mathscr{T}_h$ is a smooth bijective image of a given reference element $\hat{T}$, i.e., $T = F_T(\hat{T})$ for all $T \in \mathscr{T}_h$. Here, $\hat{T}$ is the (open) unit simplex or the (open) unit hypercube in $\mathbb{R}^d$. For $p \in \mathbb{N}$, we denote by $P_p(\hat{T})$ the set $\{\hat{x}^\alpha : 0 \leqslant \alpha_i, 0 \leqslant \sum_{i=1}^d \alpha_i \leqslant p\}$ on a simplex $\hat{T}$ or $\{\hat{x}^\alpha : 0 \leqslant \alpha_i \leqslant k, 1 \leqslant i \leqslant p\}$ on the unit hypercube $\hat{T}$ and define

$$X_h^p = \{v \in C(\bar{\Omega}) \mid v|_T \circ F_T \in P_p(\hat{T})\ \forall T \in \mathscr{T}_h\}. \tag{13}$$

We introduce *conforming* FE spaces on $\mathscr{T}_h$ for velocity and pressure, respectively, by

$$\mathbf{V}_h^r := [H^1_0(\Omega)\cap X_h^r]^d, \quad \mathbf{Q}_h^s := L^2_0(\Omega)\cap X_h^s \tag{14}$$

with r, $s \in \mathbb{N}$ and we set $\mathbf{W}_h^{r,s} := \mathbf{V}_h^r \times \mathbf{Q}_h^s$. Clearly, other conforming discrete spaces for the velocity and the pressure can be chosen (e.g., enriched with bubble functions). Moreover, for brevity, we will not present possible extensions to non-conforming methods.

A key point in the analysis of some methods is local inverse inequalities on $T \in \mathscr{T}_h$ with a constant μ_{inv} depending only on the shape-regularity